Poisson Coordinates

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Abstract—Harmonic functions are continuous maps from an open subset of \( \mathbb{R}^n \) and its boundary, \( \Omega \cup \partial \Omega \), to \( \mathbb{R} \); they are twice continuously differentiable in \( \Omega \) and satisfy the Laplace’s equation:

\[
\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \ldots + \frac{\partial^2 u}{\partial x_n^2} = 0.
\]

From a variational perspective, harmonic functions are the critical points of the following Dirichlet energy functional, which has been frequently used in gradient-domain image processing [1], [2], [3]:

\[
E_D[w] = \frac{1}{2} \int_{\Omega} |\nabla w|^2 \, dx,
\]

where \( \nabla w \) is the gradient vector field of \( w \).

Harmonic functions are also closely related to conformal geometry [4], [5], and play an important role in shape-preserving geometric computation, e.g., surface parameterization [6], [7], space deformation [8], [9], and quadrilateral remeshing [10].

In practice, we often face the problem to determine a harmonic function on a certain domain under conditions of fixed, continuous boundary values. This is known as the Dirichlet problem [11], and was well studied in the early 20th century. The existence and uniqueness of the solution as long as the domain’s boundary contains no irregular point were proved by Hilbert [12] and Perron [13]. Furthermore, the solution to the Dirichlet problem is known to be a linear combination of boundary values in the following integral form:

\[
u(x) = \int_{\partial \Omega} \mathcal{H}_\Omega(x, \zeta) f(\zeta) \, d\sigma(\zeta),
\]

where \( d\sigma(\zeta) \) is the area element of \( \partial \Omega \) at the boundary point \( \zeta \), and \( f(\zeta) \) are the given boundary values, and \( \mathcal{H}_\Omega(x, \zeta) \) are called the harmonic coordinates of \( x \) (in continuous form) with respect to each boundary point \( \zeta \). These harmonic coordinates are also closely related to Green’s functions (refer to [14]).

To study the Dirichlet problem in the discrete case, we pay attention to simplicial polytopes (i.e., the polytopes whose facets are all simplices). For such polytopes, values are given at vertices, and all other boundary values are interpolated linearly within each facet. In such a case, (1) can be rewritten as:

\[
u(x) = \sum_{v \in V(P)} \mathcal{H}_P(x, v) f(v),
\]

where \( V(P) \) are the vertices of polytope \( P \), and \( \mathcal{H}_P(x, v) \) are the harmonic coordinates of \( x \) (in discrete form) with respect to each vertex \( v \). Readers may refer to DeRose et al. [15], [16] for further details on discrete harmonic coordinates.

Harmonic coordinates are uniquely determined by the domain \( \Omega \) (or \( P \) in discrete case), but generally they do not have a closed-form expression except for a few specialized domains (e.g., \( n \)-dimensional balls). This inspires a number of numerical solvers to the Dirichlet problem [15], [16], [17], [18]. Both of them require to solve large sparse linear equations on certain global structures of the whole interior region.

As well as numerical methods, coordinates-based approaches have been proposed for fast transfinite interpolation. These techniques provide explicit expressions of harmonic-like functions with desired boundary values. As a representative instance, Floater’s Mean Value coordinates (MVCs) [19], [20], [21], [22], [23] always produce smooth interpolations in both continuous and discrete cases. However, a crucial problem of MVCs is that they...
do not produce rigorous harmonic functions on the most ordinary regions: $n$-dimensional balls.

In this work, we propose Poisson coordinates, an extension of MVCs. By allowing the projection spheres of MVCs to be translated and using the Poisson integral formula instead of the Circumferential Mean Value Theorem, we derive Poisson coordinates in continuous form, as a novel transfinite interpolation scheme. We then prove that Poisson coordinates are barycentric and also possess the same smoothness and kernel positivity as MVCs. Moreover, by choosing the projection spheres adaptively according to the positions of target points, Poisson coordinates are able to reproduce harmonic functions on $n$-dimensional balls. This important characteristic is called the pseudo-harmonic property (see [24]). MVCs do not have this feature. Finally, we give an explicit formula for discrete Poisson coordinates on 2D polygons. Experimental results reveal that Poisson coordinates have lower Dirichlet energies than MVCs on a number of typical 2D domains, particularly convex domains.

Contributions
To the best of our knowledge, Poisson coordinates are the first transfinite interpolation technique to satisfy all of the following key properties:

- Poisson coordinates are explicit coordinates in both continuous case and 2D discrete case.
- Poisson coordinates inherit MVCs’ linear precision, smoothness, and kernel positivity.
- Poisson coordinates are pseudo-harmonic.

Overview
In Section 2, we will first review the coordinates-based transfinite interpolation methods. In Section 3, we will introduce the Poisson integral formula and then use it to derive the Poisson coordinates in continuous form. In Section 4, we will further present an explicit formula for discrete Poisson coordinates on 2D polygons. Experimental results will be displayed in Section 5.

2 BACKGROUND AND PREVIOUS WORK
Previous work on coordinates-based transfinite interpolation is copious. Here we briefly review the best known approaches, with the emphasis on MVCs. Readers may refer to Belyaev’s survey [24] for a detailed analysis of these coordinates in continuous form. For the discrete case, Ju et al. [25] provided an in-depth geometric explanation for general barycentric coordinates on convex simplicial polytopes.

Mean Value Transfinite Interpolation
MVCs were first established by Floater [19], where both an interpolation formula and the corresponding explicit discrete coordinates on 2D polygons were presented.

To estimate the value at an arbitrary point $x$ within the kernel of $\Omega$, a unit sphere $S_x$ with center at $x$ (the projection sphere) is constructed. For each point $\xi$ on $S_x$, the ray $[x, \xi]$ crosses $\partial \Omega$ at a unique point $\zeta$, as shown in Fig. 1.

![Fig. 1. MVCs computation: $S_x$ is a unit sphere. $\xi$ is on $S_x$, $\zeta$ is on $\partial \Omega$. $\alpha_{x, \zeta}$ is the intersection angle between the ray $[x, \zeta]$ and the outward normal of $\partial \Omega$ at $\zeta$.](image)
with the MVCs in continuous form:
\[
\mathcal{MV}_\Omega(x, \zeta) = \frac{1}{\Phi_\Omega(x)} \frac{\cos \alpha_{x,\zeta}}{|x - \zeta|^n},
\] (7)

where:
\[
\Phi_\Omega(x) = \int_{\zeta \in \partial \Omega} \frac{\cos \alpha_{x,\zeta}}{|x - \zeta|^n} d\sigma(\zeta).
\]

Hormann and Floater [22] noticed that, the formula (6) can be extended to the whole interior region, but not limited to the kernel. In fact, for \( x \) in \( \Omega \) but outside the kernel, \( [x, \xi] \) may have multiple intersections with \( \partial \Omega \): \( \zeta_j (j = 1, \ldots, n(x, \xi)); n(x, \xi) \) odd. In such a case, we can rewrite the Mean Value formula (4) as:
\[
\begin{align*}
u(x) &= \frac{1}{\omega_n-1} \int_{\xi \in S_x} \left( \sum_{j=1}^{n(x,\xi)} (-1)^{j-1} u_x(\xi) \right) d\sigma(\xi). \\
\end{align*}
\] (8)

Substituting the term \( u_x(\xi) \) in (8) with:
\[
u_x(\xi) = u(x) + \frac{1}{|x - \zeta_j|} [f(\zeta_j) - u(x)],
\]
we obtain:
\[
u(x) = \frac{1}{\Phi_\Omega(x)} \int_{\xi \in S_x} \left( \sum_{j=1}^{n(x,\xi)} (-1)^{j-1} \frac{f(\zeta_j)}{|x - \zeta_j|} \right) d\sigma(\xi),
\] (9)

where:
\[
\Phi_\Omega(x) = \int_{\xi \in S_x} \left( \sum_{j=1}^{n(x,\xi)} (-1)^{j-1} \frac{f(\zeta_j)}{|x - \zeta_j|} \right) d\sigma(\xi).
\]

By rewriting the right hand side of (9) as an equivalent integral over \( \partial \Omega \), we can also obtain the same formula as in (6). Therefore, (6) is a transfinite interpolation.

Properties of MVCs are further discussed in [22] and [23]; it has been proved that MVCs are barycentric and smooth in 2D. In higher dimensions, similar properties have also been considered in [26].

In the discrete case, Floater [19] provided an explicit expression for discrete MVCs on 2D polygons, by calculating the integral in (6) by parts. For higher dimensions, Ju et al. [20] and Floater et al. [21] independently formulated explicit discrete MVCs on 3D simplicial polyhedrons.

More recently, Lipman et al. [27] sought coordinates \( MV(x, \zeta) \) which are zero for all \( \zeta \) invisible from \( x \), and called the results positive MVCs (PMVCs). However, PMVCs lack \( C^1 \) continuity.

**Wachspress-Warren Transfinite Interpolation**

Wachspress [28] discovered a barycentric interpolation formula on 2D polygons with affine-invariant discrete coordinates. Later, a geometric expression in terms of angles was given by Meyer et al. [29]. More recently, Warren et al. [30], [31] extended these coordinates to higher dimensions, not only for polytopes but also for general domains as a transfinite interpolation. However, Wachspress-Warren coordinates do not possess the kernel positivity.

**Laplace Transfinite Interpolation**

Schaefer et al. [32] described the Laplace transfinite interpolation. They minimized the Dirichlet energy of the piecewise linear function (3) to estimate the value at arbitrary points within the kernel. The discrete form of Laplace transfinite interpolation corresponds exactly to the well-known discrete harmonic coordinates (or cotangent weights) [33], [34], [35], which are widely used in finite element analysis and discrete differential geometry.

**Pseudo-Harmonic Transfinite Interpolation**

A transfinite interpolation is called pseudo-harmonic if it reproduces harmonic functions on \( n \)-dimensional balls. Belyaev [24] showed that none of the above methods (i.e., the Mean Value, Wachspress-Warren, and Laplace transfinite interpolations) are pseudo-harmonic. In contrast, Shepard transfinite interpolation [36] is known to be pseudo-harmonic [37]. However, Shepard interpolation is not barycentric, which greatly limits its usability. Another pseudo-harmonic, barycentric scheme is Gordon-Wixom transfinite interpolation [38], whose 2D discrete form was first derived by Belyaev [24]. However, computing 2D discrete Gordon-Wixom coordinates requires prior division of all the polygon’s sides, which might greatly enlarge the polygon’s degree, especially for non-convex polygons. Furthermore, the explicit expression for 2D discrete Gordon-Wixom coordinates is complex: Belyaev notes they are “too lengthy to present” [24]. Recently, Manson et al. proposed the positive Gordon-Wixom coordinates [39] with explicit expression for both 2D polygons and B-splines. However, positive Gordon-Wixom coordinates are not pseudo-harmonic.

**Other Transfinite Interpolation Approaches**

Several other coordinates-based transfinite interpolation techniques have also been proposed. Maximum Entropy coordinates [40] are always non-negative and smooth, but an explicit expression is not available, and a numerical calculation is required. Moving Least Square coordinates [41] can handle non-closed and self-intersecting polygons, and can reproduce polynomial functions to an arbitrary specified degree, but the computational cost is high. More importantly, neither of these methods is pseudo-harmonic.

### 3 Continuous Poisson Coordinates

To improve MVCs, we first introduce the Poisson integral formula (in Section 3.1). Based on this formula, we then present Poisson transfinite interpolation with Poisson coordinates in continuous form (in Section 3.2 and 3.3). We prove that Poisson coordinates are barycentric and pseudo-harmonic.

#### 3.1 Poisson Integral Formula

Consider the ball \( B_r \) in \( \mathbb{R}^n \) centered at the origin with radius to \( v \). There is an analytic solution to the Dirichlet
problem on $B_r$ via the following Poisson integral formula (see [14]):

$$u(x) = \int_{\zeta \in \partial B_r} \frac{r^2 - |x|^2}{r \omega_{n-1}|x - \zeta|^n} f(\zeta) d\sigma(\zeta), \quad (10)$$

where $\omega_{n-1}$ is the surface area of the unit sphere in $\mathbb{R}^n$. Note that the Circumferential Mean Value Theorem is the special case of (10) when $x$ is at the center.

By comparison to (1), this Poisson integral formula gives a closed-form expression for continuous harmonic coordinates on $B_r$:

$$H_{B_r}(x, \zeta) = \frac{r^2 - |x|^2}{r \omega_{n-1}|x - \zeta|^n}, \quad (11)$$

which is also called the Poisson kernel.

From (7) and (11), we can deduce that:

$$\mathcal{M}V_{B_r}(x, \zeta) = \Lambda_{B_r}(x) \cos \alpha_{x,\zeta} H_{B_r}(x, \zeta), \quad (12)$$

where

$$\Lambda_{B_r}(x) = \frac{1}{\Phi_{B_r}(x)} \frac{r \omega_{n-1}}{r^2 - |x|^2}$$

is a positive factor relies on $B_r$ and $x$.

From (12) we can easily see that:

**Proposition 1.** For $x \in B_r$, MVCs are equivalent to harmonic coordinates if and only if $x$ is at the center.

**Proof:** First, note that the coordinates $\mathcal{M}V_{B_r}(x, \zeta)$ and $H_{B_r}(x, \zeta)$ are equivalent if and only if all the $\alpha_{x,\zeta}$ are the same. However for the point $\zeta$ for which the ray $[x, \zeta]$ crosses the center, $\alpha_{x,\zeta}$ is exactly 0. Therefore, all the angles $\alpha_{x,\zeta}$ being the same means that they are all equal to 0, which implies that $x$ is exactly at the center of $B_r$. $\square$

This proposition reveals that MVCs are not pseudo-harmonic. However we will show in the next subsection that they can be modified to be pseudo-harmonic if the projection spheres are translated appropriately.

### 3.2 Poisson Transfinite Interpolation

To formulate Poisson transfinite interpolation, we follow the notation in Section 2. But unlike MVCs, the center of $S_x$ is not required to be $x$ any more, and we now denote it by $\kappa_x$, as shown in Fig. 2.

Like MVCs, we retain the linear estimation in (3), but replace the Mean Value formula (4) with the following equation from the Poisson integral formula [14]:

$$u(x) = \frac{1 - |x - \kappa_x|^2}{\omega_{n-1}} \int_{\xi \in S_x} \frac{1}{|x - \xi|^n} u_x(\xi) d\sigma(\xi). \quad (13)$$

Note that:

$$u(x) = \frac{1 - |x - \kappa_x|^2}{\omega_{n-1}} \int_{\xi \in S_x} \frac{1}{|x - \xi|^n} u(x) d\sigma(\xi),$$

thus (13) can be rewritten as:

$$\int_{\xi \in S_x} \frac{1}{|x - \xi|^n} (u_x(\xi) - u(x)) d\sigma(\xi) = 0. \quad (14)$$

Substituting the $u_x(\xi)$ in (14) with (3), we obtain the following Poisson interpolation:

$$u(x) = \frac{1}{\Psi_{\Omega,S_x}(x)} \int_{\xi \in S_x} \frac{f(\xi)}{|x - \xi||x - \xi|^n-1} d\sigma(\xi), \quad (15)$$

where

$$\Psi_{\Omega,S_x}(x) = \int_{\xi \in S_x} \frac{1}{|x - \xi||x - \xi|^n-1} d\sigma(\xi)$$

is a positive factor relies on $\Omega$, $x$, and $S_x$.

Going a step further, note that:

$$d\sigma(\xi) = \frac{\cos \alpha_{x,\xi}}{\cos \beta_{x,\xi}} \frac{|x - \xi|^{n-1}}{|x - \xi|^{n-1}} d\sigma(\zeta),$$

where $\beta_{x,\xi}$ is the intersection angle between $[x, \xi]$ and $[\kappa_x, \xi]$, thus we can rewrite (15) as the following Poisson transfinite interpolation (it is also for $x$ outside the kernel of $\Omega$, like MVCs, as discussed in Section 2):

$$u(x) = \frac{1}{\Psi_{\Omega,S_x}(x)} \int_{\zeta \in \partial \Omega} \frac{\cos \alpha_{x,\xi}}{\cos \beta_{x,\xi}} \frac{1}{|x - \zeta|^{n}} f(\zeta) d\sigma(\zeta), \quad (16)$$

with the Poisson coordinates in continuous form:

$$\mathcal{P}_{\Omega,S_x}(x, \zeta) = \frac{1}{\Psi_{\Omega,S_x}(x)} \frac{\cos \alpha_{x,\xi}}{\cos \beta_{x,\xi}} \frac{1}{|x - \zeta|^{n}}, \quad (17)$$

where

$$\Psi_{\Omega,S_x}(x) = \int_{\zeta \in \partial \Omega} \frac{\cos \alpha_{x,\xi}}{\cos \beta_{x,\xi}} \frac{1}{|x - \zeta|^{n}} d\sigma(\zeta).$$

Since $\beta_{x,\xi}$ and $\alpha_{x,\xi}$ are all acute for $x$ within the kernel of $\Omega$, it follows that the Poisson coordinates in (17) have kernel positivity. However, like MVCs, Poisson coordinates may be negative for points outside the kernel.

Next, we prove that Poisson coordinates have linear precision (i.e., they are barycentric):

**Proposition 2.** Poisson coordinates have linear precision.
Definition 1. Let \( \kappa \) centers adaptively place the projection spheres (i.e., to place the interpolation. In this subsection, we present a strategy to injection spheres. However, these spheres surely affect the son coordinates do not depend on the positions of pro-

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3.3 Placement of Projection Spheres

Note that kernel positivity and linear precision of Poisson coordinates do not depend on the positions of projection spheres. However, these spheres surely affect the interpolation. In this subsection, we present a strategy to adaptively place the projection spheres (i.e., to place the centers \( \kappa \)), and then prove that Poisson coordinates have linear precision. □

Proposition 4. With regular placement, if the given bound-

Proof: Assume \( f \) is a linear function defined over the whole of \( \mathbb{R}^n \). We show that Poisson transfinite interpolation reproduces \( f \). This assertion is based on two observations:

Firstly, (15) is the unique solution to (3) and (13).

Secondly, let \( u(x) = f(x) \) and \( u_\alpha(\xi) = f(\xi) \), we show that they precisely satisfy (3) and (13) at the same time. In fact, (3) is satisfied because \( f(x) \) is linear on the ray \( [x, \xi] \). Also, (13) is satisfied because \( f \) is harmonic, so Poisson integral formula reproduces \( f \). It follows that \( u(x) = f(x) \) is a solution to (3) and (13).

Therefore, (15) must be exactly equal to \( f(x) \) due to the uniqueness of the solution, which implies Poisson coordinates have linear precision. □

3.3 Placement of Projection Spheres

Fig. 4. Translate the angle \( \beta_{x,\xi} \) along \( [x, \xi] \) to locate its vertex on the base sphere. Note that \( \beta_{x,\xi} = \beta_{x,\eta} \), due to the homothety.

where \( \beta_{x,\xi} = \beta_{x,\eta} \). Since \( S \) is a fixed sphere, \( \beta_{x,\xi} (= \beta_{x,\eta}) \) is analytic, which guarantees the smoothness of Poisson coordinates:

Proposition 3. With regular placement, Poisson coordinates in (17) are smooth (\( C^\infty \)) in \( \Omega \).

Proof: With regular placement, both of \( \cos \alpha_{x,\xi} \cos \beta_{x,\xi} \) and \( |x-\zeta|^n \) are analytic. Therefore, the Poisson coordinates in (17) are composed of analytic functions and thus are \( C^\infty \) in \( \Omega \). □

More than the interior smoothness, regular placement also guarantees the continuity of the interpolation on the boundary \( \partial \Omega \):

Proposition 4. With regular placement, if the given boundary values \( f \) are continuous on \( \partial \Omega \), and the Condition 1 in Appendix is satisfied, then the function \( u \) in (16) converges to \( f \) at the boundary \( \partial \Omega \) (i.e., \( u \) interpolates \( f \)).

Proof of Proposition 4 can be found in the Appendix, and readers may skip it without loss of continuity.

By choosing different base spheres, there is a family of regular placements. However, we next focus on the following unique basic regular placement:

Definition 2. Let \( S_\Omega \) be the smallest sphere that covers \( \Omega \).

The regular placement with respect to \( S_\Omega \) is said to be the basic regular placement.

We prove that Poisson coordinates using the basic regular placement are pseudo-harmonic:

Proposition 5. With basic regular placement, Poisson coordinates are pseudo-harmonic.

Proof: Let \( \Omega \) be the \( n \)-dimensional ball \( B_r \). With basic regular placement, \( S_{B_r} \) is \( B_r \) itself, so in this case, we have \( \alpha_{x,\xi} = \beta_{x,\eta} = \beta_{x,\xi} \). Thus, the Poisson coordinates in (17) can be rewritten as the Poisson kernel in (11) multiplied by a positive factor. Hence they are equivalent, which implies that Poisson coordinates are pseudo-harmonic. □

So far, we have shown that Poisson coordinates pos-

Fig. 3. Regular placement: \( S \) is the ‘base sphere’ with center at \( \kappa \) and radius to \( r \). \( S_x \) is a unit projection sphere, whose center \( \kappa_x \) lies on the ray \( [x, \kappa] \) and satisfies \( \kappa_x = x + (\kappa - x)/r \), so that \( S_x \) and \( S \) are homothetic, and \( x \) is their homothetic center. \( S_y \) and \( S_z \) are similar.

Regular placement is an important concept. With regular placement, the angle \( \beta_{x,\xi} \) can be translated along \( [x, \xi] \) to locate its vertex on the base sphere (see Fig. 4,}\n
as shown in Fig. 3. \( S \) is said to be the base sphere of this regular placement.

\[
\kappa_x = x + (\kappa - x)/r, \quad (18)
\]

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4 DISCRETE POISSON COORDINATES

Suppose \( P \) is a simplicial polytope in \( \mathbb{R}^n \), with vertices \( v_1, v_2, ..., v_m \) and facets \( F_1, F_2, ..., F_h \). To formulate discrete Poisson coordinates on \( P \), we divide the integral in (15) into parts by facets:

\[
I(F_j) = \delta_{F_j} \int_{\xi \in S_{F_j}, \zeta \in F_j} \frac{f(\zeta)}{|x - \zeta||x - \xi|^{n-1}} d\sigma(\xi),
\]

where \( \delta_{F_j} \) is a sign that depends on \( x \) and \( F_j \); \( \delta_{F_j} \) equals 1 if \( x \) lies on the opposite side of \( F_j \)’s plane with respect to \( F_j \’s \) outward normal; otherwise \( \delta_{F_j} \) equals \(-1\). \( S_{F_j} \) is the intersection of \( S_x \) and the interior of pyramid \( x-F_j \) (for an example of \( S_{F_j} \), see the red arc in Fig. 5).

Note that \( I(F_j) \) is exactly 0 if \( x \) lies in the plane that contains \( F_j \), so we will focus on the case that \( x \) is not coplanar with \( F_j \). Suppose the boundary values \( f(\zeta) \) are linear interpolated in the simplex facet \( F_j \), which can be written as:

\[
f(\zeta) = \sum_{k:v_k \in F_j} (\zeta - x) \cdot \frac{\mu_{F_j,v_k}}{(v_k - x) \cdot \mu_{F_j,v_k}} f(v_k),
\]

where \( \cdot \) denotes the inner product of vectors; \( \mu_{F_j,v_k} \) is a normal vector to the lateral facet of pyramid \( x-F_j \) opposite vertex \( v_k \).

Substituting (20) into (19), we obtain:

\[I(F_j) = \sum_{k:v_k \in F_j} \Upsilon_{F_j} \cdot \omega_{F_j,v_k} f(v_k),\]

where:

\[\Upsilon_{F_j} = \int_{\xi \in S_{F_j}} \frac{\xi - x}{|\xi - x|^n} d\sigma(\xi)\]

is an \( n \)-dimensional vector for each facet \( F_j \), and:

\[\omega_{F_j,v_k} = \begin{cases} 0 & \text{if } x \text{ is coplanar with } F_j, \\ \delta_{F_j} \frac{(v_k - x) \cdot \mu_{F_j,v_k}}{(v_k - x) \cdot \mu_{F_j,v_k}} & \text{otherwise}. \end{cases}\]

Substituting (21) into (15), we rewrite Poisson transfinite interpolation as:

\[u(x) = \sum_{j=1}^{h} \sum_{v_k \in F_j} \omega_{F_j,v_k} \Upsilon_{F_j} \cdot \omega_{F_j,v_k} f(v_k),\]

with discrete Poisson coordinates:

\[P_{P,S_x}(x,v_k) = \frac{\sum_{j=1}^{h} \sum_{v_k \in F_j} \omega_{F_j,v_k} \Upsilon_{F_j} \cdot \omega_{F_j,v_k}}{\sum_{j=1}^{h} \sum_{v_k \in F_j} \Upsilon_{F_j} \cdot \omega_{F_j,v_k}}.\]

The greatest challenge to computing discrete Poisson coordinates in (25) is to calculate the \( \Upsilon_{F_j} \). We will next give a closed-form expression for \( \Upsilon_{F_j} \) for the case of \( n = 2 \). What is more, for 2D polygons, each vertex \( v_k \) has exactly two neighboring facets (sides) \( F_j (v_k \in F_j) \), thus the discrete Poisson coordinates are in closed-form before normalization (i.e., the numerators in (25) are, as for MVCs).

4.1 Discrete Poisson Coordinates on 2D Polygons

For \( n = 2 \), we denote the arc \( S_{F_j} \) by \( \tilde{\gamma}_{j+1} \) (we may as well suppose it is in counter-clockwise).

We next provide a closed form expression for \( \Upsilon_{F_j} \). To easy calculate the integral on the right hand side of (22), we move to the complex plane and endue each symbol with the meaning of its corresponding complex number (i.e., each 2D vector \((a,b)\) is interpreted as a complex number \(a+ib\)). We can now show that:

Proposition 6. For \( n = 2 \), \( \Upsilon_{F_j} \) in (22) can be written as:

\[\Upsilon_{F_j} = \begin{cases} i(\xi_j - \xi_{j+1}) & \text{if } x = \kappa_x, \\ i(\tau_x - \kappa_x) \log \frac{\xi_{j+1} - \tau_x}{\xi_j - \tau_x} & \text{if } x \neq \kappa_x, \end{cases}\]

where \( i \) is the imaginary unit; \( \tau_x = \kappa_x + (x - \kappa_x)/|x - \kappa_x|^2 \) is the inverse of \( x \) with respect to the circle \( S_x \), as shown in Fig. 5. The logarithm function is: \( \log z = \ln |z| + i \arg z \) with principle argument \( \arg z \in (-\pi, \pi] \).

Proof: Note that (26) is translation-invariant (i.e., it is unchanged by adding a constant to all variables). Thus we may as well suppose that \( \kappa_x = 0 \).

Using \( d\sigma(\xi) = \frac{d\xi}{i\xi} \) and \( \frac{\xi - x}{|\xi - x|^2} = \frac{1}{1/\xi - \pi} \), where \( \pi \) is the conjugate complex of \( x \), we can write \( \Upsilon_{F_j} \) in (22) as the following complex integral:

\[\Upsilon_{F_j} = -i \int_{\xi_j}^{\xi_{j+1}} \frac{1}{1 - \pi} d\xi.\]

Clearly, (27) is equal to \( i(\xi_j - \xi_{j+1}) \) when \( x = 0 \). For \( x \neq 0 \), (27) is equal to:

\[\Upsilon_{F_j} = i \tau_x \int_{\xi_j}^{\xi_{j+1}} \frac{1}{\xi - \tau_x} d\xi,
\]

where \( \tau_x = 1/\pi \) is the inverse of \( x \) with respect to \( S_x \). Note that \( \tau_x \) is outside the circle \( S_x \), therefore the winding number of \( S_x \) around \( \tau_x \) is exactly zero, and hence
the integral \( \int_{\xi_j}^{\xi_{j+1}} \frac{1}{\xi - x} d\xi \) in (28) equals \( \log \frac{\xi_{j+1} - x}{\xi_j - x} \), completing the proof of Proposition 6. \( \square \)

After presenting the closed form expression for \( \mathcal{Y}_{F_j} \), we next calculate the 2D discrete Poisson coordinates. We first rewrite the numerators of (25) as:

\[
P_{P,S_2}(x,v_j) = \mathcal{Y}_{F_j} \cdot \omega_{F_j,v_j} + \mathcal{Y}_{F_{j-1}} \cdot \omega_{F_{j-1},v_j},
\]
where \( F_j \) is the side \([v_j,v_{j+1}]\), and all the indices are interpreted modulo \( m \) (i.e., \( v_0 = v_m \) and \( v_{m+1} = v_1 \)). \( P_{P,S}(x,v_j) \) are called the unnormalized discrete Poisson coordinates. From them, discrete Poisson coordinates can be obtained via:

\[
P_{P,S}(x,v_j) = P_{P,S_2}(x,v_j) / \sum_k P_{P,S_2}(x,v_k). \tag{30}
\]

Now, let us consider the term \( \mathcal{Y}_{F_{j+1}} \cdot \omega_{F_{j+1},v_j} \) in (29). Note that \( \omega_{F_{j+1},v_j} \) equals 0 if \( x \) is colinear with side \([v_j,v_{j+1}]\), which implies \( \mathcal{Y}_{F_{j+1}} \cdot \omega_{F_{j+1},v_j} \) is also equal to 0. So, we next assume that \( x \) is not colinear with \([v_j,v_{j+1}]\). Using (23), we can have:

\[
\mathcal{Y}_{F_j} \cdot (\mu_{F_j,v_j}|v_{j+1}-x|) = |\mathcal{Y}_{F_j} \times (v_{j+1} - x)|,
\]
where \( \times \) denotes the cross product of vectors, and:

\[
\delta_{F_j}(v_j \times x \cdot \mu_{F_j,v_j}|v_{j+1}-x|) = 2A_{\Delta x v_j v_{j+1}},
\]
where \( A_{\Delta x y z} \) denotes the signed area of triangle \( XYZ \). Therefore, the unnormalized 2D discrete Poisson coordinates in (29) can be rewritten as follow (suppose that \( x \) is not colinear with side \([v_j,v_{j+1}]\) either):

\[
P_{P,S_2}(x,v_j) = \frac{|\mathcal{Y}_{F_j} \times (v_{j+1} - x)|}{2A_{\Delta x v_j v_{j+1}}} + \frac{|\mathcal{Y}_{F_{j-1}} \times (v_{j-1} - x)|}{2A_{\Delta x v_{j-1} v_j}}. \tag{31}
\]

Note that when \( x = \kappa x \), the vector \( \mathcal{Y}_{F_j} = i(\xi_j - \xi_{j+1}) \) lies along the bisector of \( \angle v_j x v_{j+1} \), and its length equals \( |\xi_j - \xi_{j+1}| = 2\sin(\angle v_j x v_{j+1}/2) \). In this case, we have:

\[
\frac{2A_{\Delta x v_j v_{j+1}}}{2A_{\Delta x v_j v_{j+1}}} = \frac{2|v_{j+1} - x| \sin^2(\angle v_j x v_{j+1}/2)}{|v_{j+1} - x||v_j - x| \sin \angle v_j x v_{j+1}} = \frac{1}{|v_j - x|} \tan \frac{\angle v_j x v_{j+1}}{2},
\]
and hence (31) can be rewritten as:

\[
P_{P,S_2}(x,v_j) = \frac{1}{|v_j - x|} \left( \tan \frac{\angle v_j x v_{j+1}}{2} + \tan \frac{\angle v_{j-1} x v_j}{2} \right),
\]
which corresponds exactly to MVCs.

Pseudocode for 2D discrete Poisson coordinates computation is summarized in Algorithm 1.

---

5 RESULTS AND DISCUSSION

We demonstrate Poisson coordinates (with basic regular placement) on various 2D polygons in Fig. 7, and compare them to both discrete harmonic coordinates and MVCs. Note that both MVCs and Poisson coordinates are smooth. To illustrate that Poisson coordinates are closer to discrete harmonic coordinates than MVCs, we place them to both discrete harmonic coordinates and MVCs in Table 1, and further show their differences to discrete harmonic coordinates in the last 2 columns of Fig. 7.

Table 1 contains the Dirichlet energies of discrete harmonic coordinates (denoted by \( E_0 \)), Poisson coordinates (denoted by \( E_1 \)), and MVCs (denoted by \( E_2 \)), for the polygons in Fig. 7. These energies are calculated using a
Fig. 7. The left 3 columns illustrate the results of discrete harmonic coordinates, Poisson coordinates (with basic regular placement), and MVCs for 2D polygons; contour lines for values 0.75, 0.50, 0.25, 0.10, 0.03, 0.005 are in black. The right 2 columns show the DIFFERENCES between either Poisson coordinates or MVCs and discrete harmonic coordinates. Dirichlet energies of the left 12 sub-figures are shown in Table 1.

Other than the good performance of harmonic approximation, another key feature of Poisson coordinates is that they allow users to define their own placements for projection spheres for different purposes. In this paper we have described the regular placement, particularly the basic regular placement. Note that MVCs can also be considered to be Poisson coordinates with a regular placement, where the radius of the base sphere is infinity. The technique of PMVCs [27] could also be applied to Poisson coordinates, however that is beyond the scope of this paper. A weak point of our approach is that Poisson coordinates with a regular placement can only be extended to the interior region of the base sphere, but MVCs can be extended to the entire space, since their base sphere is infinite.

6 SUMMARY AND FUTURE WORK

In this paper we introduced Poisson coordinates, and gave explicit formulae for Poisson coordinates in both
continuous and 2D discrete cases. Poisson coordinates have MVCs’ kernel positivity and linear precision, and allow placing the projection spheres at will. We further introduced the concept of regular placement, and particularly basic regular placement, using which the Poisson coordinates are smooth and pseudo-harmonic. We also derived an explicit formula for discrete Poisson coordinates on 2D polygons. Experiments showed that Poisson coordinates are ‘more harmonic’ than MVCs on a set of typical 2D convex polygons.

We believe this work will provide useful ideas for further studies on barycentric coordinates and transfinite interpolation. On the theoretical end, it would be useful to give an explicit formula for discrete Poisson coordinates in higher dimensions (i.e., to give an explicit formula for the $T_F$ in (22) for $n \geq 3$). On the practical end, it is also possible to build alternative Poisson coordinates using different placement of the projection spheres, which may help provide desired boundary behaviors or overcome other restrictions.

Finally, the authors conjecture that:

**Conjecture 1.** With basic regular placement, Poisson coordinates always have lower Dirichlet energies than MVCs, for arbitrary 2D convex domains.

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**Appendix**

**Proof of Proposition 4**

Suppose $f$ is continuous on the boundary $\partial \Omega$. We prove that, with a regular placement, the function $u$ described in (16) converges to $f$ at the boundary $\partial \Omega$ (i.e., $u$ really interpolates $f$). Following similar proofs for MVCs in [23] and [26], we first rewrite (16) as:

$$u(x) = \frac{1}{\Psi_{\Omega,S} (x)} \int_{\xi \in S} \frac{dA_{x,\xi}}{\cos \beta_{x,\xi}} \sum_{j=1}^{n(x,\xi)} \left( -1 \right)^{j-1} \frac{1}{|x - \zeta_j|} f(\zeta_j), \quad (32)$$

where $dA_{x,\xi}$ is an element of solid angle in $\mathbb{R}^n$; $\zeta_j (j = 1, \ldots, n(x,\xi))$ are the intersections between the ray $[x, \xi)$ and $\partial \Omega$, and:

$$\Psi_{\Omega,S} (x) = \int_{\xi \in S} \frac{dA_{x,\xi}}{\cos \beta_{x,\xi}} \prod_{j=1}^{n(x,\xi)} \frac{-1}{|x - \zeta_j|}.$$

As in [23], we prove two key inequalities:

$$\Psi_{\Omega,S} (x) \geq \frac{C_1}{d(x,\Omega)}, \quad (33)$$

$$\int_{\xi \in S} \frac{dA_{x,\xi}}{\cos \beta_{x,\xi}} \prod_{j=1}^{n(x,\xi)} \frac{1}{|x - \zeta_j|} \leq C_2 \Psi_{\Omega,S} (x), \quad (34)$$

with positive constants $C_1$ and $C_2$; $d(x,\Omega)$ is the distance from $x$ to $\Omega$. Note that the only difference between our inequalities and the ones in [23] is the additional term $\cos \beta_{x,\xi}$.

Finally, for $\Psi_{\Omega,S} (x)$ to be true because:

$$\Psi_{\Omega,S} (x) \geq \int_{\xi \in S} \frac{dA_{x,\xi}}{\cos \beta_{x,\xi}} \sum_{j=1}^{n(x,\xi)} (-1)^{j-1} \frac{1}{|x - \zeta_j|} \geq \frac{C_1}{d(x,\Omega)},$$

where the first inequality is based on $\cos \beta_{x,\xi} \leq 1$, and the second inequality is proved in [23] and [26].

To prove (34), we additionally require the region $\Omega$ to satisfy the following condition:

**Condition 1.** There is a positive constant $\Delta$, such that for any $x$ and $\xi$, either $\cos \beta_{x,\xi} \geq \Delta$ or $n(x,\xi) = 1$.

Note that $\cos \beta_{x,\xi} < \Delta$ implies that $x$ should be very close to the base sphere $S$, and the ray $[x, \xi)$ should be nearly perpendicular to the segment $[\xi, x)$ (see Fig. 8).

In this case, Condition 1 asks $[x, \xi)$ to cross $\partial \Omega$ exactly once.

In fact, Condition 1 is easy to satisfy. For many common domains (e.g., convex domains and 2D polygons), Condition 1 is satisfied for all possible $S$. For arbitrary domains, Condition 1 is also satisfied if the base sphere $S$ does not touch $\Omega$.

With Condition 1, let $I_x = \{ \xi : \xi \in S, \cos \beta_{x,\xi} \geq \Delta \}$ and $J_x = \{ \xi : \xi \in S, \cos \beta_{x,\xi} < \Delta \}$.

Then, as proved in [23] and [26]:

$$\int_{\xi \in I_x} \frac{dA_{x,\xi}}{\cos \beta_{x,\xi}} \sum_{j=1}^{n(x,\xi)} \frac{1}{|x - \zeta_j|} \leq \frac{C_1}{\Delta} \int_{\xi \in I_x} \frac{dA_{x,\xi}}{\cos \beta_{x,\xi}} \sum_{j=1}^{n(x,\xi)} \frac{1}{|x - \zeta_j|} + \int_{\xi \in J_x} \frac{dA_{x,\xi}}{\cos \beta_{x,\xi}} \frac{1}{|x - \zeta_j|}$$

$$\leq \frac{C_2}{\Delta} \int_{\xi \in I_x} \frac{dA_{x,\xi}}{\cos \beta_{x,\xi}} \sum_{j=1}^{n(x,\xi)} (-1)^{j-1} \frac{1}{|x - \zeta_j|} + \int_{\xi \in J_x} \frac{dA_{x,\xi}}{\cos \beta_{x,\xi}} \frac{1}{|x - \zeta_j|} \leq \max \left( \frac{C_2}{\Delta}, 1 \right) \Psi_{\Omega,S} (x),$$

Fig. 8. Condition 1: Whenever $\beta_{x,\xi} = \beta_{x,\eta}$ is close to a right-angle, $x$ should be close to $S$ and $[x, \xi)$ should be nearly perpendicular to $[\xi, x)$. In this case, we ask $[x, \xi)$ to cross only one boundary point $\zeta_1 \in \partial \Omega$. 

which implies (34).

Using (33) and (34), the rest of the proof follows that of Theorem 4 of [23].

REFERENCES


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