Conversion of a triangular Bézier patch into three rectangular Bézier patches

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Abstract

In this paper, we give an explicit formula for converting a triangular Bézier patch into three non-degenerate rectangular Bézier patches of the same degree. The use of certain operators simplifies the formulation of such a decomposition. The formula yields a stable recursive algorithm for computing the control points of the rectangular patches. We also illustrate the formula with an example.

Keywords: Rectangular Bézier patches; Triangular Bézier patches; Parametric transformation

1. Introduction

Bézier surfaces are among the most commonly used parametric surface in CAGD (Barnhill, 1985; Farin, 1990). Bézier surfaces are defined in terms of Bernstein polynomials: the univariate Bernstein polynomial \( B_i^n (s) = \binom{n}{i} s^i (1 - s)^{n-i} \) and the bivariate Bernstein polynomial \( B_{i,j,k}^n (u,v,w) = \binom{n}{i,j,k} u^i v^j w^k \). A triangular Bézier patch of degree \( n \) with control vertices \( T_{i,j,k} \) is represented by

\[
T(u,v,w) = \sum_{i+j+k=n} T_{i,j,k} B_{i,j,k}^n (u,v,w), \quad u, v, w \geq 0, \quad u + v + w = 1,
\]

and a rectangular Bézier patch of degree \( m \times n \) with control vertices \( P_{ij} \) is represented by

\[
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\[ P(s, t) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{ij} B_i^m(s) B_j^n(t), \quad 0 \leq s, t \leq 1. \] (2)

Since they have different Bernstein bases, triangular and rectangular Bézier patches have fundamentally different geometric properties, e.g., their Bézier nets have very different structures. Their incompatibility causes difficulties when both kinds of patches are used in the same CAD system. Thus the conversion of one type into another has aroused interest. Goldman and Filip (1987) present an explicit formula for the conversion from a rectangular Bézier patch of degree \(m \times n\) into two triangular Bézier patches of degree \(m + n\). Brueckner (1980) converts a triangular Bézier patch to a trimmed surface of a rectangular Bézier patch. However, trimmed surfaces are not too suitable in surface modeling. Hu (1993) gives a corner cutting algorithm for representing a triangular Bézier patch as a degenerate integral rectangular Bézier patch, but degeneracy is not a desirable property (Farin, 1989).

Recently, Lino and Wilde (1991) proposed a formula for decomposing a triangular Bézier patch into three non-degenerate rectangular subpatches of the same degree. But there are two points of concern:

- The rectangular subpatches are rational even if the original triangular Bézier surface is polynomial.
- An algorithm for computation cannot be derived directly from the formula.

The objective of this paper is to present a stable algorithm to convert a triangular Bézier patch into three rectangular Bézier patches of the same degree. Furthermore, these rectangular patches are non-degenerate and polynomial. An explicit formula of this conversion is given in Section 2. In Section 3, based on the conversion formula, we write a recursive algorithm for computing the control vertices of the rectangular patches. This algorithm is stable. For illustration, the conversion of a triangular patch of degree 3 is given in Section 4.

2. Conversion formula

Consider a non-degenerate triangular Bézier patch given as in (1) defined on a domain triangle \(D\). We divide \(D\) into three quadrilaterals. Choose points \(P_1, P_2,\) and \(P_3\), one from each edge of \(D\), in which the vertices of \(D\) are avoided. Then pick an interior point \(P\) of the triangle \(P_1P_2P_3\). The segments \(PP_1, PP_2, PP_3\) divide \(D\) into three quadrilaterals \(D_1, D_2, \) and \(D_3\). See Fig. 1(a). The curves correspond to those segments divide the triangular Bézier patch into three rectangular patches.

We first give a representation of the rectangular patch defined on \(D_1\) as Bézier patch and show that it is non-degenerate. The patches correspond to \(D_2\) and \(D_3\) can be similarly dealt with. We then show that the boundaries of these three patches fit together nicely.

Without loss of generality, we may choose \(D = \{(u, v) : u, v \geq \epsilon; u + v \leq 1\}\) as shown in Fig. 1(b). For convenience, we write \(T_{ij}\) for \(T_{i,j,n-i-j}\) and write \(B_{ij}^n(u, v)\) for \(B_{i,j,n-i-j}^n(u, v, 1 - u - v)\). Then the patch (1) can be rewritten as
Fig. 1. Decomposition of domain triangle.

\[ T(u, v) = \sum_{i+j=0}^{n} T_{ij} B_{ij}^n(u, v), \quad u, v \geq 0, \quad u + v \leq 1. \]  (3)

To simplify notation and the sequence of deduction, we shall make use of the following operators.

- The invariant operator \( I: \Pi T_{ij} = T_{ij} \).
- The shifting operators \( E_i: E_i T_{ij} = T_{i+1,j}, \ E_2 T_{ij} = T_{i,j+1} \).
- The difference operators \( \Delta_i: \Delta_i T_{ij} = T_{i+1,j} - T_{ij}, \ \Delta_2 T_{ij} = T_{i,j+1} - T_{ij} \).

Using these operators, the triangular Bézier patch (3) can be represented by

\[ T(u, v) = (u E_1 + v E_2 + (1 - u - v) I)^n T_{00} = (\Delta_1 u + \Delta_2 v + I)^n T_{00}. \]  (4)

A rectangular Bézier representation of the rectangular patch defined on \( D_1 \) can now be stated. Note that the operators greatly facilitate the computation in the proof of the following theorem.

**Theorem 1.** The trimmed surface of \( T(u, v) \) defined on domain \( D_1 \) can be represented as an integral rectangular Bézier patch of degree \( n \times n \) whose control vertices \( P_{ij} \) are determined by

\[ P_{ij} = \sum_{k=0}^{i} \sum_{l=0}^{n-i} \binom{i}{k} \binom{n-i}{l} \binom{n}{j} Q_{kl}^{(i)} \quad 0 \leq i, j \leq n, \]  (5)

in which for \( 0 \leq i \leq n, \ 0 \leq k \leq i, \ 0 \leq l \leq n - i, \)

\[ Q_{kl}^{(i)} = (a E_1 + c E_2 + (1 - a - c) I)^k (b E_1 + (1 - b) I)^l d E_2 + (1 - d) I)^l T_{00} \]  (6)

where \((a, c), (0, d), (b, 0)\) are the coordinates of \( P, P_1 \) and \( P_3 \) respectively.

**Proof.** The parametric transformation
\begin{align*}
\begin{cases}
  u = (a - b) st + bs, \\
v = (c - d) st + dt
\end{cases}
\end{align*}
\tag{7}

converts the quadrilateral \( D_1 \) into square \([0, 1] \times [0, 1] \).

Substituting (7) into (4), the following computation shows that the trimmed surface of \( T(u, v) \) defined on \( D_1 \) is indeed an integral Bézier patch of degree \( n \times n \).

\[
T(u, v) = (\Delta_1 u + \Delta_2 v + l)^n T_{00}
\]

\[
= ((a - b) \Delta_1 st + b \Delta_1 s + (c - d) \Delta_2 st + d \Delta_2 t + l)^n T_{00}
\]

\[
= ((a - b) \Delta_1 st + b \Delta_1 s(t + (1 - t)) + (c - d) \Delta_2 st
\]

\[
+ d \Delta_2 t(s + (1 - s)) + l(s + (1 - s))(t + (1 - t)))^n T_{00}
\]

\[
= (s((I + h \Delta_1)(1 - t) + (I + a \Delta_1 + c \Delta_2)t)
\]

\[
+ (1 - s)(I(1 - t) + (I + d \Delta_2)t))^n T_{00}
\]

\[
= \sum_{i=0}^{n} B_i^n(s)(((I + h \Delta_1)(1 - t) + (I + a \Delta_1 + c \Delta_2)t)^i
\]

\[
\times (I(1 - t) + (I + d \Delta_2)t)^n T_{00}
\]

\[
= \sum_{i=0}^{n} B_i^n(s) \sum_{k=0}^{i} \binom{n}{k} B_k^{n-i}(t)((I + a \Delta_1 + c \Delta_2)^k
\]

\[
\times (I + h \Delta_1)^{i-k}(I + d \Delta_2)^l T_{00}
\]

\[
= \sum_{i=0}^{n} B_i^n(s) \sum_{k=0}^{i} \binom{n}{k} \binom{n-i}{k+i} B_k^{n-i}(t)(I + a \Delta_1 + c \Delta_2)^k
\]

\[
\times (I + h \Delta_1)^{i-k}(I + d \Delta_2)^l T_{00}
\]

\[
= \sum_{i=0}^{n} \sum_{k=0}^{i} \left( \binom{i}{k} \binom{n-i}{k+i} \right) B_i^n(s) B_k^{n-i}(t) Q_{kl}^{(i)}
\]

\[
= \sum_{i=0}^{n} \sum_{j=0}^{n} P_{ij} B_i^n(s) B_j^n(t). \quad \square
\]

**Theorem 2.** If the original triangular Bézier patch is non-degenerate, so is the rectangular Bézier patch corresponding to \( D_1 \).

**Proof.** By the chain rule, we have

\[
\left( \frac{\partial T}{\partial \xi} \right) = \left( \frac{\partial u}{\partial \xi} \frac{\partial v}{\partial \eta} \right) \left( \frac{\partial T}{\partial u} \frac{\partial T}{\partial v} \right) = \left( \begin{array}{cc}
(a - b) t + b & (c - d) t \\
(a - b) s & (c - d) s + d
\end{array} \right) \left( \frac{\partial u}{\partial \xi} \frac{\partial v}{\partial \eta} \right) \quad \tag{8}
\]

Since the original triangular Bézier patch is non-degenerate, \( \frac{\partial u}{\partial \xi} \times \frac{\partial v}{\partial \eta} \neq 0 \) for \((u, v) \in D_1 \).

We first show that the Jacobian matrix \( J \) of transformation (7) is non-singular. From (8), we see that the determinant of \( J, |J| = b(c - d)s + a(a - b)t + bd \). Because \( P \) is an interior point of triangle \( P_1 P_2 P_3 \), we have from Fig. 2
Fig. 2. Construction associated with $|J|$.

$$
\frac{d - c}{d} s + \frac{b - a}{b} t = \frac{|P_1 G_1|}{|P_1 O|} s + \frac{|H_1 P_3|}{|O P_3|} t = \frac{|P_1 G_2|}{|P_1 P_3|} s + \frac{|H_2 P_3|}{|P_1 P_3|} t.
$$

Since $(s, t) \in [0, 1] \times [0, 1]$, therefore

$$
\frac{d - c}{d} s + \frac{b - a}{b} t \leq \frac{|P_1 G_2|}{|P_1 P_3|} + \frac{|H_2 P_3|}{|P_1 P_3|} < 1.
$$

Thus $|J| = b(c - d) s + d(a - b) t + b d > 0$. So $J$ is non-singular. Together with $\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \neq 0$ for $(u, v) \in D_1$, it follows from (8) that $\frac{\partial r}{\partial s} \times \frac{\partial r}{\partial t} \neq 0$ for $(s, t) \in [0, 1] \times [0, 1]$. Hence, the subpatch defined on $D_1$ is non-degenerate. □

We claim that the boundaries of the three rectangular Bézier patches fit together nicely. We will only consider the common boundary between the two Bézier patches that correspond to $D_1$ and $D_2$. From (5), we have

$$
P_{n j} = Q^{(n)}_{k, 0} = (a E_1 + c E_2 + (1 - a - c) I)^j (b E_1 + (1 - b) I)^{n - j} T_{00}
$$

It is clear that the control vertices of the boundary determined by $P_{03}$ of the subpatch defined on $D_1$ depend only on the coordinates of $P$ and $P_3$. The same is also true for the subpatch defined on $D_2$. Thus their control vertices along their common boundary coincide.

3. Algorithm

It is obvious from (6) that we have a recursive algorithm to compute $Q^{(i)}_{k, l}$. Not so obvious is the fact that we can also derive from (5) a recursive algorithm to compute $P_{ij}$. From the identity

$$
\sum_{k=0}^{i} \binom{i}{k} A_k = \sum_{k=0}^{i-1} \binom{i-1}{k} A_k + \sum_{k=0}^{i-1} \binom{i-1}{k} A_{k+1}
$$
we have

\[ P_{ij} = \sum_{k=0}^{i} \binom{i}{k} \binom{n-i}{j-k} Q_{k,j-k}^{(i)} \]

\[ = \sum_{k=0}^{i-1} \binom{i-1}{k} \binom{n-i}{j-k} Q_{k,j-k}^{(i)} + \sum_{k=0}^{i-1} \binom{i-1}{k} \binom{n-i}{j-(k+1)} Q_{k+1,j-(k+1)}^{(i)} \]

\[ = \frac{n-j}{n} \sum_{k=0}^{i} \binom{i-1}{k} \binom{n-i}{j-k} Q_{k,j-k}^{(i)} + \frac{j}{n} \sum_{k=0}^{i-1} \binom{i-1}{k} \binom{n-i}{j-(k+1)} Q_{k+1,j-(k+1)}^{(i)}. \tag{9} \]

We are now in a position to give an explicit algorithm to compute the control vertices of the rectangular Bézier patch corresponding to \( D_1 \).

Conversion Algorithm.

1. Calculation of \( Q_{kl}^{(i)} \) \( (0 \leq i \leq n, \ 0 \leq k \leq i \ 0 \leq l \leq n-i) \).

   For \( i = 0, 1, \ldots, n \),
   - Set \( g_{kl}^{(i)} = T_{kl} \ k = 0, 1, \ldots, i, \ l = 0, 1, \ldots, n-i \)
   - For \( k = 0, 1, \ldots, i \)
     - For \( l = 0, 1, \ldots, n-i \)
       - For \( r = 1, \ldots, i \)
         - For \( h = 0, 1, \ldots, i-r \)
           - If \( r \leq k \)
             - \( g_{hl}^{(r)} = a g_{h+1,l}^{(r-1)} + c g_{h,l+1}^{(r-1)} + (1-a-c) g_{h,l}^{(r-1)} \)
           - If \( r \geq k+1 \)
             - \( g_{hl}^{(r)} = (1-b) g_{h,l}^{(r-1)} + b g_{h+1,l}^{(r-1)} \)
         - Next \( h \)
       - Next \( r \)
     - Next \( l \)
   - Next \( k \)
   - Next \( i \)

2. Calculation of \( P_{ij} \).

   For \( i = 0, 1, \ldots, n \)
   - For \( j = 0, 1, \ldots, n \)
     - set \( f_{k,j-k}^{(0)} = Q_{k,j-k}^{(i)} \ k = 0, 1, \ldots, i \)
   - While \( (i \geq 1) \)
     - For \( r = 1, \ldots, i \)
For \( k = 0, 1, \ldots, i - r \)
\[
f_{k,j-k}^{(r)} = \frac{n-i+r-(j-k)}{n-i+r} f_{k,j-k}^{(r-1)} + \frac{j-k}{n-i+r} f_{k+1,j-(k+1)}^{(r-1)}
\]
Next \( k \)
Next \( r \)
\[
P_{ij} = f_{00}^{(i)}
\]
Next \( j \)
Next \( i \)

The domain triangle \( D \) of the triangular Bézier patch (3) is \( \{(u,v) ; u, v \geq 0 ; u + v \leq 1 \} \). Since \( P_1 = (0,d) \) and \( P_3 = (b,0) \) are not vertices of \( D \); and \( P = (a,c) \) is an interior point of \( D \), we have \( 0 \leq a, b, c, d \leq 1 \) and \( 0 \leq 1 - a - c \leq 1 \). Hence all the points that appear in the above algorithm, such as \( g_{d}^{(r)} \) and \( f_{k,j-k}^{(r)} \), are convex linear combination of previous points. We thus conclude that the algorithm is stable.

4. Example

We give the rectangular Bézier representation of the rectangular patch defined on \( D_1 \) for a triangular Bézier patch of degree 3. The most reasonable choices of \( a, b, c, d \) are \( a = c = \frac{1}{3} \), \( b = d = \frac{1}{2} \), which have the merit of giving a well-balanced partition. We denote the groups of operators that appear in the expression of \( Q_{d}^{(i)} \) in (6) by

\[
A = aE_1 + cE_2 + (1 - a - c)I = \frac{1}{3} (E_1 + E_2 + I),
B_1 = bE_1 + (1 - b)I = \frac{1}{2} (E_1 + I),
B_2 = dE_2 + (1 - d)I = \frac{1}{2} (E_2 + I).
\]

From equations (5) and (6), we obtain
\[
\begin{pmatrix}
P_{00} & P_{01} & P_{02} & P_{03} \\
P_{10} & P_{11} & P_{12} & P_{13} \\
P_{20} & P_{21} & P_{22} & P_{23} \\
P_{30} & P_{31} & P_{32} & P_{33}
\end{pmatrix}
= \begin{pmatrix}
I & B_2 & B_2^2 & B_2^3 \\
B_1 & \frac{1}{2} B_1 B_2 + \frac{1}{3} A & \frac{1}{4} B_1 B_2^2 + \frac{1}{3} A B_2 & \frac{1}{6} B_1 B_2^3 + \frac{1}{3} A^2 B_2 \\
B_1^2 & \frac{1}{3} B_1^2 B_2 + \frac{2}{3} A B_1 & \frac{1}{3} B_1 B_2^2 + \frac{1}{3} A^2 & \frac{1}{3} A^2 B_2 \\
B_1^3 & A B_1^2 & A^2 B_1 & A^3
\end{pmatrix}
\times T_{00}
\]

A direct computation yields the following control vertices. Note that in full notation, \( T_{ij} = T_{i,j,n-i-j}, \)

\[
P_{00} = T_{00},
\]
\[
P_{01} = \frac{1}{3} (T_{01} + T_{00}),
\]
\[
P_{02} = \frac{1}{4} (T_{02} + 2T_{01} + T_{00}),
\]
\[
P_{03} = \frac{1}{8} (T_{03} + 3T_{02} + 3T_{01} + T_{00}),
\]
\[
P_{10} = \frac{1}{3} (T_{10} + T_{00}),
\]
\[
P_{11} = \frac{1}{18} (3T_{11} + 5T_{10} + 5T_{01} + 5T_{00}),
\]
\[
P_{12} = \frac{1}{72} (3T_{12} + 14T_{11} + 11T_{10} + 11T_{02} + 22T_{01} + 11T_{00}),
\]
\[
P_{13} = \frac{1}{180} (5T_{13} + 15T_{12} + 30T_{11} + 45T_{10} + 45T_{03} + 70T_{02} + 70T_{01} + 35T_{00}),
\]
\[
P_{20} = \frac{1}{8} (T_{20} + T_{10} + T_{00}),
\]
\[
P_{21} = \frac{1}{24} (T_{21} + 3T_{20} + 3T_{11} + T_{10} + 3T_{02} + 3T_{01} + T_{00}),
\]
\[
P_{22} = \frac{1}{48} (T_{22} + 3T_{21} + 6T_{20} + 9T_{12} + 9T_{11} + 6T_{10} + 3T_{03} + 6T_{02} + 6T_{01} + T_{00}),
\]
\[
P_{23} = \frac{1}{96} (T_{23} + T_{22} + T_{21} + T_{20} + T_{13} + T_{12} + T_{11} + T_{10} + T_{03} + T_{02} + T_{01} + T_{00}),
\]
\[
P_{30} = \frac{1}{96} (T_{30} + T_{20} + T_{10} + T_{00}),
\]
\[
P_{31} = \frac{1}{288} (T_{31} + T_{21} + T_{11} + T_{02} + T_{01} + T_{00}),
\]
\[
P_{32} = \frac{1}{576} (T_{32} + T_{22} + T_{12} + T_{03} + T_{02} + T_{01} + T_{00}),
\]
\[
P_{33} = \frac{1}{1152} (T_{33} + T_{23} + T_{13} + T_{04} + T_{03} + T_{02} + T_{01} + T_{00}).
\]
\[
P_{13} = \frac{1}{12} (T_{12} + 2T_{11} + T_{10} + T_{03} + 3T_{02} + 3T_{01} + T_{00}),
\]
\[
P_{20} = \frac{1}{4} (T_{20} + 2T_{10} + T_{00}),
\]
\[
P_{21} = \frac{1}{12} (3T_{21} + 11T_{20} + 14T_{11} + 22T_{10} + 11T_{01} + 11T_{00}),
\]
\[
P_{22} = \frac{1}{24} (3T_{21} + 5T_{20} + 3T_{12} + 13T_{11} + 10T_{10} + 5T_{02} + 10T_{01} + 5T_{00}),
\]
\[
P_{23} = \frac{1}{18} (T_{21} + T_{20} + 2T_{12} + 4T_{11} + 2T_{10} + T_{03} + 3T_{02} + 3T_{01} + T_{00}),
\]
\[
P_{30} = \frac{1}{8} (T_{30} + 3T_{20} + 3T_{10} + T_{00}),
\]
\[
P_{31} = \frac{1}{12} (T_{30} + T_{21} + 3T_{20} + 2T_{11} + 3T_{10} + T_{01} + T_{00}),
\]
\[
P_{32} = \frac{1}{18} (T_{30} + 2T_{21} + 3T_{20} + T_{12} + 4T_{11} + 3T_{10} + T_{02} + 2T_{01} + T_{00}),
\]
\[
P_{33} = \frac{1}{27} (T_{30} + 3T_{21} + 3T_{20} + 3T_{12} + 6T_{11} + 3T_{10} + T_{03} + 3T_{02} + 3T_{01} + T_{00}).
\]

By symmetry, to obtain the Bézier representation of the other two rectangular subpatches, we only have to replace $T_{kj}$ by $T_{n-i-j,i,j}$ or $T_{i,n-i-j,j}$ in the results above.

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**References**


