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Generalized Subdivision of Bézier Surfaces

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In this paper, subdivision methods for rectangular Bézier surfaces are generalized to subdivide a rectangular Bézier surface patch of degree $n \times m$ into two rectangular Bézier surface patches of degree $n \times (m + n)$, while the parameter domain of the Bézier surface is decomposed into two trapezoids. As an application, a conversion from rectangular Bézier surfaces to triangular Bézier surfaces is presented.

1. INTRODUCTION

The Bézier surface is a very useful tool in surface modeling [1, 2]. Subdivision algorithms for Bézier surfaces are very important in rendering and curve–curve, curve–surface, and surface–surface intersection calculations [3, 4]. It is well known that, subdivision algorithms for rectangular Bézier surfaces, which decompose a rectangular patch into two rectangular patches of the same degree, splitting its parametric domain into two rectangles, are based on the subdivision of Bézier curves. If we split the parametric domain of a rectangular Bézier surface into two trapezoids, the surface is again decomposed into two surface patches. Can these two trimmed surface patches be represented as rectangular Bézier surface patches? How do we obtain the new control points from those of the original surface patch?

These questions are considered in the second section. By using parameter transformations, we show that the control points of these two rectangular Bézier patches can be obtained from those of the original surface patch. A corner-cutting algorithm is developed in the third section, which decomposes a rectangular Bézier patch of degree $n \times m$ into two rectangular patches of degree $n \times (m + n)$.

The paper concludes with an application using the generalized subdivision algorithm to split a rectangular Bézier patch of degree $n \times m$ into two triangular Bézier patches of degree $m + n$.

2. MAIN RESULTS

A rectangular Bézier surface of degree $n \times m$ can be represented by

$$P(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} P_{ij} B_i^n(u) B_j^m(v), \quad 0 \le u, v \le 1,$$

where $B_i^n(u) = \binom{n}{i} u^i (1 - u)^{n-i}$ are univariate Bernstein polynomials of degree n, and $P_{ij} (0 \le i \le n, 0 \le j \le m)$ are control points of $P(u, v)$. Without loss of generality, we consider the trimmed surface patch defined on the domain D_1 (see Fig. 1).

First of all, we introduce some operator symbols for a Bézier curve $P(t)$ with control points $P_i (0 \le i \le n)$.

1. The invariant operator I: $IP_i = P_i$,
2. The shifting operator E: $EP_i = P_{i+1}$,
3. The difference operator Δ: $\Delta P_i = (E - I) P_i = P_{i+1} - P_i$,
4. The degree elevation operator A_n:

$$A_n = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ \frac{1}{n} & \frac{n-1}{n} & 0 & \cdots & 0 & 0 \\ 0 & \frac{2}{n} & \frac{n-2}{n} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \frac{n-1}{n} & \frac{1}{n} \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}_{(n+1)\times n} . \quad (1)$$

More details about E, Δ, and $A_q (q = n + 1, n + 2, \ldots$ for Bézier curves of q degree) can be found in [5–7].

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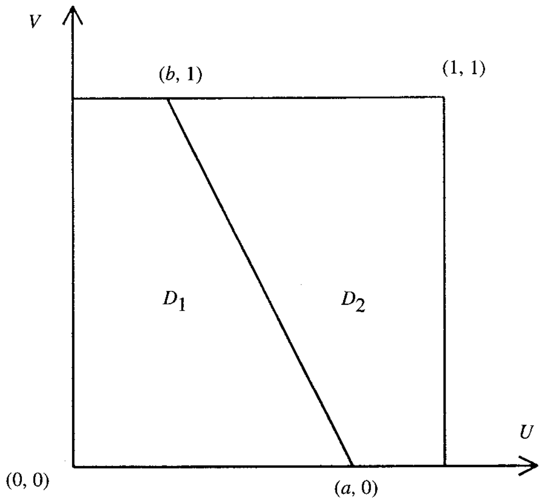

FIG. 1. Decomposition of the domain.

With the help of the above notation, we have

$$P(t) = \sum_{i=0}^{n} B_i^n(t) P_i = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} E^i P_0 = ((1-t)I + tE)^n P_0 = (I + t\Delta)^n P_0. \tag{2}$$

Similarly, we introduce operator symbols E_1, E_2, Δ_1, Δ_2 for the rectangular Bézier surface $P(u, v)$ by

$$E_1 P_{ij} = P_{i+1,j}, \quad \Delta_1 P_{ij} = P_{i+1,j} - P_{ij},$$
$$E_2 P_{ij} = P_{i,j+1}, \quad \Delta_2 P_{ij} = P_{i,j+1} - P_{ij}.$$

Then $P(u, v)$ can be represented by

$$P(u, v) = (I + u\Delta_1)^n (I + v\Delta_2)^m P_{00} \tag{3}$$

By using the parameter transformation

$$T: \begin{cases} u = s(bt + a(1-t)) \\ v = t \end{cases}$$

(see Fig. 2), we have the following results.

Lemma 1. $(I + st\Delta)^n P_0 = \sum_{i=0}^{n} B_i^n(s)(I + t\Delta)^i P_0$.

Proof.

$$(I + st\Delta)^n P_0 = ((1-s)I + s(I + t\Delta))^n P_0 = \sum_{i=0}^{n} B_i^n(s)(I + t\Delta)^i P_0. \quad ■$$

Lemma 2. $(I + (bt + a(1-t))\Delta_1)^i P_{0j} = \sum_{k=0}^{i}((1-a)I + a\mathrm{E}_1)^{i-k}((1-b)I + bE_1)^k B_k^i(t) P_{0j}$.

Proof.

$$\begin{aligned}
&(I + (bt + a(1-t))\Delta_1)^i P_{0j} \\
&= ((1 - bt - a(1-t))I + (bt + a(1-t))E_1)^i P_{0j} \\
&= (((1-a)(1-t) + (1-b)t)I + (bt + a(1-t))E_1)^i P_{0j} \\
&= (((1-a)I + aE_1)(1-t) + ((1-b)I + bE_1)t)^i P_{0j} \\
&= \sum_{k=0}^{i} ((1-a)I + aE_1)^{i-k}((1-b)I + bE_1)^k B_k^i(t) P_{0j}. \quad ■
\end{aligned}$$

Theorem 3. *The trimmed surface patch of $P(u, v)((u, v) \in D_1)$ can be represented as a rectangular Bézier surface $\tilde{P}(s, t)$ of degree $n \times (m + n)$, and its control points $\tilde{P}_{ij}(0 \le i \le n, 0 \le j \le m + n)$ are determined by*

$$\begin{pmatrix} \tilde{P}_{i0} \\ \tilde{P}_{i1} \\ \vdots \\ \tilde{P}_{i,m+n} \end{pmatrix} = A_{m+n} A_{m+n-1} \cdots A_{m+i+1} \begin{pmatrix} H_{i0} \\ H_{i1} \\ \vdots \\ H_{i,m+i} \end{pmatrix}, \quad i = 0, 1, \ldots, n, \tag{4}$$

where

$$Q_{kj}^{(i)} = ((1-a)I + aE_1)^{i-k}((1-b)I + bE_1)^k P_{0j} \quad \begin{matrix} i = 0, 1, \ldots, n \\ j = 0, 1, \ldots, m \\ k = 0, 1, \ldots, i, \end{matrix} \tag{5}$$

$$H_{il} = \sum_{\substack{j=0 \\ j+k=l}}^{m} \sum_{k=0}^{i} \frac{\binom{m}{j}\binom{i}{k}}{\binom{m+i}{j+k}} Q_{kj}^{(i)} = \sum_{k=0}^{i} \binom{i}{k} \frac{\binom{m}{l-k}}{\binom{m+i}{l}} Q_{k,l-k}^{(i)} \quad \begin{matrix} i = 0, 1, \ldots, n \\ l = 0, 1, \ldots, m + i. \end{matrix} \tag{6}$$

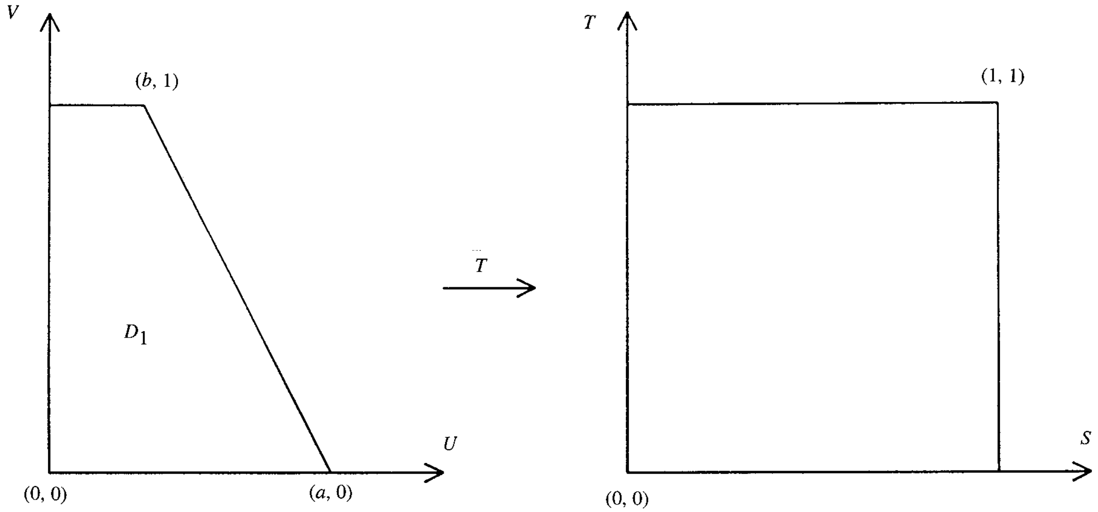

FIG. 2. Transformation of the domain.

Proof.

$$\begin{aligned}
P(u,v) &= (I+u\Delta_1)^n(I+v\Delta_2)^m P_{00}\\
&= (I+s(bt+a(1-t))\Delta_1)^n(I+t\Delta_2)^m P_{00}\\
&= \sum_{i=0}^{n} B_i^n(s)(I+(bt+a(1-t))\Delta_1)^i(I+t\Delta_2)^m P_{00}\\
&= \sum_{i=0}^{n} B_i^n(s)\sum_{k=0}^{i}((1-a)I+aE_1)^{i-k}\\
&\qquad ((1-b)I+bE_1)^k B_k^i(t)\sum_{j=0}^{m} B_j^m(t)P_{0j}\\
&= \sum_{i=0}^{n} B_i^n(s)\sum_{k=0}^{i}\sum_{j=0}^{m} B_k^i(t)B_j^m(t)Q_{kj}^{(i)}\\
&= \sum_{i=0}^{n} B_i^n(s)\sum_{j=0}^{m}\sum_{k=0}^{i}\frac{\binom{m}{j}\binom{i}{k}}{\binom{m+i}{j+k}}B_{j+k}^{m+i}(t)Q_{kj}^{(i)}\\
&= \sum_{i=0}^{n}\sum_{l=0}^{m+i} H_{il}B_i^n(s)B_l^{m+i}(t).
\end{aligned}$$

By using the degree elevation operators $A_q(q = m+i+1, m+i+2, \dots, m+n)$, Eq. (4) can be obtained. ∎

3. THE SUBDIVISION ALGORITHM

In addition to the fact that Eq. (4) and (5) can lead to corner-cutting algorithms, we would like to point out that, H_{ij} can be computed by a recursive process. By use of the identity

$$\sum_{k=0}^{i}\binom{i}{k}A_k = \sum_{k=0}^{i-1}\binom{i-1}{k}A_k + \sum_{k=0}^{i-1}\binom{i-1}{k}A_{k+1}, \tag{7}$$

we have

$$\begin{aligned}
H_{il} &= \sum_{k=0}^{i}\binom{i}{k}\frac{\binom{m}{l-k}}{\binom{m+i}{l}}Q_{k,l-k}^{(i)}\\
&= \sum_{k=0}^{i-1}\binom{i-1}{k}\frac{\binom{m}{l-k}}{\binom{m+i}{l}}Q_{k,l-k}^{(i)}\\
&\quad + \sum_{k=0}^{i-1}\binom{i-1}{k}\frac{\binom{m}{l-(k+1)}}{\binom{m+i}{l}}Q_{k+1,l-(k+1)}^{(i)} \qquad (8)\\
&= \frac{m+i-l}{m+i}\sum_{k=0}^{i-1}\binom{i-1}{k}\frac{\binom{m}{l-k}}{\binom{m+i-1}{l}}Q_{k,l-k}^{(i)}\\
&\quad + \frac{l}{m+i}\sum_{k=0}^{i-1}\binom{i-1}{k}\frac{\binom{m}{l-k-1}}{\binom{m+i-1}{l-1}}Q_{k+1,l-k-1}^{(i)}.
\end{aligned}$$

Obviously, Eq. (8) shows that H_{ij} can be calculated by a recursive process. Then, a generalized subdivision algorithm for rectangular Bézier surfaces can be derived as follows.

ALGORITHM.

1) Calculation of $Q_{kj}^{(i)}$ $(0 \le i \le n, \quad 0 \le j \le m \quad 0 \le k \le i)$.

For $i = 0, 1, \ldots, n$
  Set $P_{kj}^{(0)} = P_{kj} \qquad k = 0, 1, \ldots, i, \quad j = 0, 1, \ldots, m$
  For $k = 0, 1, \ldots, i$
  For $j = 0, 1, \ldots, m$
    For $r = 1, \ldots, i$
    For $h = 0, 1, \ldots, i - r$
      If $r \le k$
      $P_{hj}^{(r)} = (1 - b)P_{hj}^{(r-1)} + bP_{h+l,j}^{(r-1)}$
      If $r \ge k + 1$
      $P_{hj}^{(r)} = (1 - a)P_{hj}^{(r-1)} + aP_{h+1,j}^{(r-1)}$
    Next h
    Next r
    $Q_{kj}^{(i)} = P_{0j}^{(i)}$
  Next j
  Next k
Next i

2) Calculation of H_{il} $(0 \le i \le n, \quad 0 \le l \le m + i)$.

For $i = 0, 1, \ldots, n$
For $l = 0, 1, \ldots, m + i$
  Set $f_{k,l-k}^{(0)} = Q_{k,l-k}^{(i)} \quad k = 0, 1, \ldots, i$
  For $r = 1, \ldots, i$
  For $k = 0, 1, \ldots, i - r$

$$f_{k,l-k}^{(r)} = \frac{m + r - (l - k)}{m + r} f_{k,l-k}^{(r-1)} + \frac{l - k}{m + r} f_{k+1,l-k-1}^{(r-1)}$$

  Next k
  Next r
  $H_{il} = f_{0l}^{(i)}$
Next l
Next i

3) Calculation of $\tilde{P}_{ij}$ $(0 \le i \le n, \quad 0 \le j \le m + n)$.

Set $H_{ij}^{(m+i)} = H_{ij}$ $i = 0, 1, \ldots, n; j = 0, 1, \ldots, m + i$
For $i = 0, 1, \ldots, n$
  For $q = m + i + 1, m + i + 2, \ldots, m + n$
    For $j = 0, 1, \ldots, q$

$$H_{ij}^{(q)} = \frac{j}{q} H_{i,j-1}^{(q-1)} + \frac{q - j}{q} H_{ij}^{(q-1)}$$

    Next j
  Next q
Next i
Set$\tilde{P}_{ij} = H_{ij}^{(m+n)}$ $i = 0, 1, \ldots, n; j = 0, 1, \ldots, m + n$

4. AN APPLICATION

Triangular and rectangular Bézier surface methods together form one of the main techniques for surface modeling, and it is always an interesting problem to study the internal relations between two types of Bézier surface patches. Brueckner [8] showed a conversion from triangular patches to rectangular patches which defines the triangular patch as a trimmed surface of the rectangular patch. Goldman and Filip [9] presented a conversion formula that precisely splits an integral rectangular Bézier patch of degree $n \times m$ into two integral triangular Bézier patches of degree $m + n$. As an application of the generalized subdivision algorithm presented in the third section, we show that a rectangular Bézier patch can be split into two triangular Bézier patches by a corner-cutting algorithm.

Let $a = 1$, $b = 0$, it is obvious that the rectangular patch $P(u, v)$ can be split into two degenerate rectangular patches, and it is convenient to prove that these two rectangular patches of degree $n \times (m + n)$ can be represented as triangular patches of degree $m + n$. We present an algorithm for computing the control points of these triangular patches. Without loss of generality, we consider the surface patch defined on domain D_1. A proof of this algorithm is given in the Appendix.

ALGORITHM.

1) Compute the control points $\tilde{P}_{ij}(0 \le i \le n, 0 \le j \le m + n)$ of $\tilde{P}(s, t)$ by using the subdivision algorithm presented in the previous section.

2) Calculate the control points $\hat{P}_{i,j,k}(i, j, k \ge 0, i + j + k = m + n)$ of the corresponding triangular patch $\hat{P}(u, v, w)$.

Set $g_{ij}^{(n)} = \tilde{P}_{ij}$ $i = 0, 1, \ldots, n; j = 0, 1, \ldots, m + n$
For $j = 0, 1, \ldots, m + n$
  if $j < m$
  For $q = n + 1, n + 2, \ldots, m + n - j$
    For $i = 0, 1, \ldots, q$

$$g_{ij}^{(q)} = \frac{i}{q} g_{i-1,j}^{(q-1)} + \frac{q - i}{q} g_{ij}^{(q-1)}$$

    Next i
  Next q
  If $j > m$
  For $q = n - 1, n - 2, \ldots, m + n - j$
    For $i = 0, 1, \ldots, q$

$$g_{ij}^{(q)} = \frac{q + 1}{q + 1 - i} g_{i,j}^{(q+1)} - \frac{i}{q + 1 - i} g_{i-1,j}^{(q)}$$

    Next i
  Next q
Next j
Set $\hat{P}_{i,j,k} = g_{ij}^{(m+n-j)}$ for $0 \le i + j \le m + n$

Obviously, this is a corner-cutting algorithm; however, an efficient conversion algorithm cannot be derived directly from the explicit conversion formula in [7].

APPENDIX: A PROOF OF THE CONVERSION ALGORITHM

By using the transformation

$$T^{-1}: \begin{cases} s = \dfrac{u}{1-v} = \dfrac{u}{u+w}, \\ t = v \end{cases}$$

we have

$$\begin{aligned} & B_i^{m+n-j}(s)B_j^{m+n}(t) \\ &= \binom{m+n-j}{i}\binom{m+n}{j} s^i(1-s)^{m+n-i-j}t^j(1-t)^{m+n-j} \\ &= \binom{m+n}{i,j,k}\left(\frac{u}{u+w}\right)^i\left(\frac{w}{u+w}\right)^{m+n-i-j} v^j(1-v)^{m+n-j} \\ &= B_{i,j,k}^{m+n}(u,v,w) \end{aligned}$$

$$u+v+w=1, \quad i+j+k=m+n.$$

Therefore, a rectangular Bézier patch

$$Q(s,t) = \sum_{i=0}^{m+n}\sum_{j=0}^{m+n} q_{ij}B_i^{m+n}(s)B_j^{m+n}(t), \quad 0 \le u, v \le 1,$$

degenerates into a triangular patch with parameters u and v if and only if

$$\begin{cases} \sum_{i=0}^{m+n}(-1)^i\binom{m+n}{i}q_{ik} = 0 \\ \sum_{i=0}^{m+n-1}(-1)^i\binom{m+n-1}{i}q_{ik} = 0 \\ \vdots \\ \sum_{i=0}^{m+n-k+1}(-1)^i\binom{m+n-k+1}{i}q_{ik} = 0 \end{cases} \tag{9}$$

for $k = 1, 2, \ldots, m+n$.

Obviously, using the transformation T^{-1}, the parameters of $\tilde{P}(s, t)$ turn into u and v, i.e.,

$$\begin{array}{ccccc} P(u,v) & & \tilde{P}(s,t) & & P(u,v,w) \\ n \times m \text{ degree} & \xrightarrow{T} & n \times (m+n) \text{ degree} & \xrightarrow{T^{-1}} & \le m+n \text{ degree} \\ (u,v) \in D_1 & & 0 \le u, v \le 1 & & 0 \le u, v, u+v \le 1 \end{array}$$

Since $P(u, v)((u, v) \in D_1)$ can be represented as a triangular Bézier patch of degree $m + n$, (shown in [7]), the control points of $\tilde{P}(s, t)$ must satisfy Eq. (9) for $k = m+1, m+2, \ldots, m+n$, where $q_{ik} = \tilde{P}_{ik}$, and this completes the proof.

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