Connectedness of Random Walk Segmentation *

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Abstract

Connectedness of random walk segmentation is examined, and novel properties are discovered, by considering electrical circuits equivalent to random walks. A theoretical analysis shows that earlier conclusions concerning connectedness of random walk segmentation results are incorrect, and counterexamples are demonstrated.

Keywords: Image segmentation, random walk, Laplace’s equation, counterexample, connectedness.

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1 Introduction

The random walk algorithm proposed by Grady [1] is a leading method for seeded image segmentation. In this graph-based algorithm, edge weights denote the likelihood that a random walk will cross that edge. For each pixel (node), the probability is computed of a random walk starting at it first reaching each seed in turn. These probabilities are compared, and the pixel given the same label as the seed with greatest probability. (Multiple seeds may share the same label). An important property given by Grady’s paper is that each segmented region is guaranteed to be connected to one or more seeds with that region’s label: isolated regions without seeds do not occur. This is important, as it implies that the random walk approach avoids the noisy or fragmented segmentations that can sometimes result from other algorithms. Unfortunately, as we show here, this property does not always hold. We give a counterexample in Section 2 and a theoretical analysis in Section 3.

2 Counterexample and explanation

First, we give a specific counterexample, showing that the connectedness property does not always hold. Figure 1 shows the random walk segmentation result of a color image containing 4 regions ($R_I, R_U, R_L,$ and $R_O$), starting from 4 seeds of 3 types ($s_U, s_L,$ and $s_O$). The segmentation result is consistent with the image information, where all four regions of different color are separated. However, this is a counterexample to the connectedness proposition in [1] since the region $R_I$ output by the segmentation does not contain any seed points. To see why this arises, we consider the image and seeds in more detail; readers should refer to Grady’s original paper for an explanation of the algorithm. Let $D(R_1, R_2)$ be the color difference between regions $R_1$ and $R_2$. In this image, $D(R_I, R_U) = D(R_I, R_L) \gg D(R_U, R_L) \gg D(R_U, R_O) = D(R_L, R_O)$. Further, let $U_O$ and $L_O$ be the probability that a random walk starting at the center point of $R_I$ first reaches seed type $s_O$ via region $R_U$ or $R_L$ respectively. We experimentally observe that $U_U - U_O > U_O - U_L > 0$, which also matches with intuition. Thus $U_O + L_O = 2U_O > U_U + U_L = U_U + L_U = L_L + U_L$ (equality due to symmetry), i.e. the center point should be labeled the same as $s_O$. This gives an intuitive understanding of how isolated regions containing no seeds may occur, if the user fails to place an adequate number of seeds in suitable places.
Figure 1: A counterexample: (a) input image containing 4 regions, and 4 seeds of 3 types; (b) random walk segmentation result; probability of a random walk starting at each pixel first reaching seed type $s_U$ (c) or $s_O$ (d).

3 Theoretical Analysis

We now consider where Grady’s proof of connectedness breaks down. Following his notation, let $x_i^s$ be the probability of a random walk starting at node $i$ first arriving at a seed labelled $s$. In Proposition 1 of [1], although $\forall f \neq s, \exists i$ such that $x_i^s > x_i^f$, $i$ can be different for each $f$. The proof of this proposition fails to show that any connected subset of unseeded nodes assigned to segmentation $s$ must contain at least one seed labeled $s$. (Nevertheless, this proposition does hold if there are only two seed types, which is a special result of our Observation 1 in Section 3).

One may ask whether we can make some simple amendments, such as changing the assignment rule, to fix this problem. The answer is ‘No’. To show this, we analyze the random walk problem using an equivalent electrical circuit network. We further assume that the algorithm works on general graphs, and generalize the segmentation problem to a special convex hull partition problem (Observation 1). Then we show step by step (Proposition 1-5) that for any segmented graph which satisfies the connectedness property and has more than 3 label types, even allowing the algorithm to have more general weighting (Proposition 4) and assignment rule (Proposition 5), it is still possible to replace some part of it in a way which makes the connectedness property fail. Finally, Proposition 5 provides a condition under which random walk segmentation is guaranteed to give a connected result.

According to [2], a random walk graph has an equivalent electrical circuit network with conductances equal to the edge weights, and voltage sources replacing seed points. In this circuit network, the probability $x_i^s$ is equal to the voltage at $i$ if we give the seeds labeled $s$ unit potential and other seeds zero potential. For a passive sub-network (PSN; without voltage sources) $X$ of the circuit, we denote its boundary nodes as $N(X) = \{y_k\}$. Here boundary nodes are those outside $X,
having at least one neighbor within \( X \); \( X \) comprises the nodes it contains, the
conductance associated with these nodes, and the conductance between boundary
nodes. Let the probabilities of a random walk starting at node \( p \) first reaching a
seed of type \( S = \{s_1, \cdots, s_m\} \) be a vector \( x_p = [x_p^{s_1}, \cdots, x_p^{s_m}] \).

**Observation 1.** For each node \( p \) in \( X \), \( x_p = \sum_k \lambda^k_p x_{y_k}, \lambda^k_p \geq 0, \sum_k \lambda^k_p = 1 \), and
the weights \( \lambda^k_p \) are uniquely determined by \( X \). Let the vector be \( \lambda_p = [\cdots, \lambda^k_p, \cdots] \).

Using a circuit network analogy, each PSN \( X \) can be represented by a con-
ductance matrix \( H \) (uniquely determined by \( X \); see Proposition 1 for a detailed
expression). The conductance matrix \( H \) for a circuit represents the linear relation
between the inward current and boundary voltage, \( I = HU \), where \( I \) and \( U \) are
vectors formed by concatenating inward current and boundary voltage values at
boundary nodes, and \( H \) is determined by the structure of the circuit network. If
no output nodes are connected by zero resistance (infinite conductance), \( H \) is well-
defined. To prove Proposition 3, we first analyze some properties (Proposition 1,
2) of the conductance matrix \( H \):

**Proposition 1.** Assuming there are \( M \) boundary points, \( H = \{h_{ij}\}_{M \times M} \) has the
following properties:

1. Symmetry: \( H^T = H \).
2. Zero sum: \( Hl_{M \times 1} = 0 \), where the vector \( l \) has all elements 1.
3. Sign: \( h_{ij} \geq 0 \) for \( i = j \) and \( h_{ij} \leq 0 \) for \( i \neq j \).

**Proof.** If \( C \) the \( M \times M \) Laplacian matrix for boundary nodes, \( U_X \) the inner volt-
ages, \( B \) the \( n \times M \) negative connection matrix for connections between inner nodes
and the boundary, and \( L \) is the \( n \times n \) matrix for the inner nodes, Ohm’s Law and
Kirchhoff’s Rules give

\[
I = CU - \text{diag}(B^Tl)U + B^TU_X, \quad LU_X + BU = 0
\]

Thus, \( U_X = -L^{-1}BU \). It follows that \( I = -(\text{diag}(B^Tl) - B^T L^{-1}B + C)U \). Hence:

1. \( H = -\text{diag}(B^Tl) - B^T L^{-1}B + C = -\text{diag}(B^Tl)^T - (B^T L^{-1}B)^T + C^T = H^T \).
2. Using the fact that \( Bl + B = 0, Cl = 0 \), we have \( Hl = -(\text{diag}(B^Tl) + B^T L^{-1}B)l = -B^Tl \). Thus
\( c_{ij} \geq 0 \) for \( i = j \) and \( c_{ij} \leq 0 \) for \( i \neq j \), so property 3 follows from property 2. \( \square \)
Proposition 2. For any $H$ satisfying Proposition 1, we can construct a PSN with conductance matrix $H$.

Proof. The proof is simple—we just connect boundary nodes. For boundary nodes $i$ and $j$, we add an edge with weight (conductance) $-h_{ij}$. It is easy to check that the conductance matrix is $H$. Note that no inner nodes are created, and such a matrix corresponds to $C$ in the proof of Proposition 1.

Proposition 3. Given a conductance matrix $H = \{h_{ij}\}_{M \times M}$ and a set of vectors $\{\lambda_w\}$ satisfying $\lambda_w^k \geq 0$, $\sum_k \lambda_w^k = 1$, and $(h_{ij} = 0) \Rightarrow (\lambda_w^i \cdot \lambda_w^j = 0)$ for each $i < j$, we can construct a PSN $X$ with conductance matrix $H$, containing nodes $\{w\}$ which has vectors $\{\lambda_w\}$.

Proof. For each vector $\lambda_w$, we add an inner node $w$; for each boundary node $i$, we add an edge $e_{w,i}$ between $w$ and $i$, with weight $\alpha_w \lambda_i \lambda_w$ (where $\alpha_w$ is a positive constant associated with $w$, to be determined later). Note that when $\lambda_w = 0$, the creation of $e_{w,i}$ is not needed. The conductance matrix associated with $e_{w,i}$ is $H_w = \{h_{w,ij}\}_{M \times M}$, where

$$h_{w,ij} = \begin{cases} -\alpha_w \lambda_i \lambda_w, & i \neq j \\ \sum_{k \neq i} -h_{w,ik}, & i = j. \end{cases}$$

Now, $H_w$ is a valid conductance matrix, and $\lim_{\alpha_w \to 0} H_w = 0$. Thus, it is easy to check that small enough positive numbers $\{\alpha_w\}_n$ exist such that $H_S = H - \sum_w H_w$ is still a valid conductance matrix. After constructing a PSN for $H_S$ as in Proposition 2, we get a PSN consisting of $n + 1$ parts ($n$ are created for vectors, and one is created by connecting to boundary nodes as in Proposition 2). These parts do not share nodes in the interior, so the overall conductance matrix is $\sum_w H_w + H_S = H$. Each node $w$ has vector $\lambda_w$.

From basic electrical circuit principles, if a PSN is replaced by another with the same conductance matrix, other parts of the circuit network are not affected. Proposition 3 shows that if $X$ is a connected graph, for any set of voltages inside the convex hull of voltages of $N(X)$, we can design another sub-network $X'$ to replace $X$, such that voltage and current at every node of the rest of the circuit does not change, and the nodes of $X'$ give the set of specified voltages.

Proposition 4. For a graph (circuit network), in which each node (pixel) $i$ is associated with a ‘color’ $c_i$ (scalar or vector), the weight for edge $e_{ij}$ is given by $g(c_i - c_j)$, where the continuous function $g : V \to R^+$ satisfies $g(c) = g(-c)$, and $\lim_{\|c\| \to \infty} g(c) = 0$. Given a set of vectors as in Proposition 3, we can replace a PSN $X$ by another PSN $X'$, without changing other parts of the circuit network. The new network contains nodes with the given vectors and has the same weighting.
function (edge weights are uniquely determined by the ‘color’ of each edge’s two end nodes).

Proof. We can replace $X$ by $X'$ to construct desired vectors while keeping the conductance matrix unchanged using a similar approach to that in Proposition 3. To construct ‘colors’ so that edges agree with the weighting function $g$, $c_i$ for a newly created node $i$ is firstly random initialized. For each edge $e_{ij} \neq g(c_i - c_j)$, let $w_{ij} = g(0)g(c_i - c_j) / [g(0) + g(c_i - c_j)]$. If $e_{ij} < w_{ij}$, we create another node $j'$ with color $c'_j$, remove edge $e_{ij}$, and connect $i, j'$ and $j'$. The equivalent weight is $h(c'_j) = g(c_i - c'_j)g(c'_j - c_j) + g(c'_j - c_j))$. Note that $h(c_i) = w_{ij} > e_{ij}$ and $\lim_{||c'_j|| \to \infty} g(c'_j) = 0$, so $\exists c'_j \in V$ such that $h(c'_j) = e_{ij}$, i.e. the network remains unaltered. If $e_{ij} > w_{ij}$, let $k = [e_{ij}/w_{ij}]$, remove edge $e_{ij}$ and create $k$ nodes with color $c_i$, each connected to node $i$ and $j$. If $e_{ij} - kw_{ij} > 0$, we create a node as above with weight $h(c'_j) = e_{ij} - kw_{ij}$. The total equivalent weight is just $e_{ij}$.

Proposition 4 suggests that even in normal color based graph segmentation (the edge weights are determined by ‘color’ differences between nodes), counter examples can still be easily constructed by replacing a sub-network by another.

Figure 2(a) shows a small example which meets the connectedness property, with $x^{(c,e,g)}_f = \{0.55, 0.4, 0.05\}$ and $x^{(c,e,g)}_h = \{0.05, 0.4, 0.55\}$. While $x^{(c,e,g)}_i = \{0.3, 0.4, 0.3\}$ lies inside the convex hull of $\{x_f, x_h\}$, it does not belong to the same seed as either $x_h$ or $x_f$. It can be constructed by replacing the PSN in (a) (only...
containing the edge with weight 0.5) by another according to Proposition 3. Figure 2(c), constructed according to Proposition 4, is the image part corresponding to the counterexample in Figure 2(b).

From the above, we can design the interior voltages of a connected sub network provided that they lie inside the convex hull of voltages of the boundary.

**Observation 2.** We define a segmentation to be a map \( f \) from the vector space \( \{ x_p \} \) to segmentation identifier—i.e., the segmentation is a partition of the convex hull generated by voltages at seed points. The segmentation method in [1] is thus a special case of segmentation, with \( x_{s_i} \) having \( i \)-th value 1, and other values 0, and \( f(x_p) = \arg \max_i (x^i_p) \). Note that \( f \) is ill-defined when \( \arg \max_i (x^i_p) \) has more than one value. Generally, such points have zero measure in the Euclidean space containing the convex hull generated by voltages at seed points; the convex hull has positive volume. Suppose the convex hull of seed voltages \( C \) is partitioned into \( \{ C_i \}_{1 \times m} \), with each \( C_i \) corresponding to a label type \( x_{s_i} \in C_i \), and having positive volume. For Grady’s connectivity statement to be true, it requires that, for any \( \{ s_{i_k} \} \subset S \), \( \bigcup C_{i_k} \) is convex.

**Proposition 5.** The requirement in Observation 2 cannot be satisfied for \( m \geq 3 \).

Proposition 5 shows that Grady’s proof holds only for the two-label case, in which the line segment with end points \( x_{s_1}, x_{s_2} \) is partitioned into two segments at the midpoint \((x_{s_1} + x_{s_2})/2\).

**Proof.** Let the interior of \( C_i \) be \( \bar{C}_i \) and the closure of \( C_i \) be \( \bar{C}_i \). Suppose \( \{ \bar{C}_i \}_{1 \times m} \) satisfies the requirements in Observation 2 and \( m \geq 3 \). It is easy to show that \( C_i \) is convex and \( \{ \bar{C}_i \} \) still satisfies the requirements in Observation 2. Then, according to the separation theorem of convex sets, there exist hyperplanes \( P_1, P_2 \), separating
\(C_1, C_2, C_3\) as shown in Figure 3, and \(P_i \cup C_j = \emptyset, i = 1, 2, j = 1, 2, 3\). We choose \(P_i \in C_i, i = 1, 2, 3, \) and \(K_1 = P_1 P_2 \cap P_1, K_2 = P_2 P_3 \cap P_2\), where \(P_i P_j\) denotes the line segment with end points \(P_i\) and \(P_j\). We choose a sequence, \(W_1, W_2, \ldots, W_n, \ldots\), satisfying \(W_n \in P_1 K_1, W_n \neq K_1, \) and \(W_n \to K_1\). Note that \(\bar{C}_1 \cup \bar{C}_2\) is convex, and \(W_n \in \bar{C}_1 \cup \bar{C}_2\). As \(W_n \notin \bar{C}_2\), we have \(W_n \in \bar{C}_1\). The closedness of \(\bar{C}_1\) implies that \(K_1 \in \bar{C}_1\). Similarly, \(K_2 \in \bar{C}_3\). But obviously, the interior of segment \(K_1 K_2, (\lambda K_1 + (1 - \lambda) K_2) \notin \bar{C}_1 \cup \bar{C}_3, 0 < \lambda < 1\). This contradicts the requirement of convexity of \(\bar{C}_1 \cup \bar{C}_3\). \(\square\)

4 Conclusion

We have given a counterexample to disprove Grady’s assertion concerning the connectedness of segmentations produced by random walk [1]. Further theoretical discussions show that the original assertion is not true in its most general form. Despite this deficit, experiments on many real world images do result in connected segmentations—random walk segmentation is indeed a powerful tool in many situations. Our discussion gives a new way to understand the structure of random walk segmentations, and what users can expect from random walk segmentation.

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