Computer Graphics

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Today’s Topics

- **Why splines?**
- **B-Spline Curves and properties**
- **B-Spline surfaces**
- **NURBS curves and Surfaces**

Why to introduce B-Spline (B样条)

- **Bezier curve/surface has many advantages**, but they have two main shortcomings:
  - Bezier curve/surface cannot be modified locally (局部修改).
  - It is very complex to satisfy geometric continuity conditions for Bezier curves or surfaces joining.

- **History of B-splines**
  - In 1946, Schoenberg proposed a spline-based method to approximate curves.
  - It’s motivated by runge-kutta problem in interpolation: high degree polynomial may surge upper and down
  - Why not use lower degree piecewise polynomial with continuous joining?
  - that’s Spline

- But people thought it’s impossible to use Spline in shape design, because complicated computation
- In 1972, based on Schoenberg’s work, Gordon and Riesenfeld introduced “B-Spline” and lots of corresponding geometric algorithms.
- B-Spline retains all advantages of Bezier curves, and overcomes the shortcomings of Bezier curves.

- Tips for understanding B-Spline?
  - Spline function interpolation is well known, it can be calculated by solving a tridiagonal equations(三对角方程).
  - For a given partition of an interval, we can compute Spline curve interpolation similarly.
  - All splines over a given partition will form a linear space. The basis function of this linear space is called B-Spline basis function.

– Similar to Bezier Curve using Bernstein basis functions, B-Spline curves uses B-Spline basis functions.

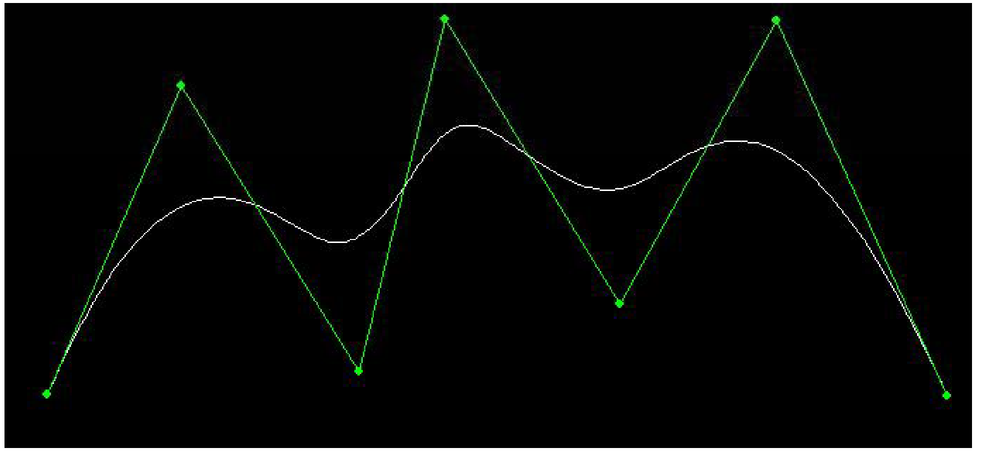
B-Spline curves and it's Properties

- Formula of B-Spline Curve. $P(t) = \sum_{i=0}^{n} P_i B_{i,n}(t), \quad t \in [0,1]$

$$P(t) = \sum_{i=0}^{n} P_i N_{i,k}(t)$$

  - $P_i (i = 0,1,\cdots,n)$ are control points.
  - $N_{i,k}(t)$ (i=0,1,..,n) are the i-th B-Spline basis function of order k. B-Spline basis function is a order k (degree k -1) piecewise polynomial (分段多项式) determined by the knot vector, which is a non-decreasing set of numbers.

- Demo of B-spline



- The story of order & degree
  - G Farin: degree, Computer Aided Geometric Design
  - Les Piegl: order, Computer Aided Design

B-Spline Basis Function

- Definition of B-Spline Basis Function
  - de Boor-Cox recursion formula:

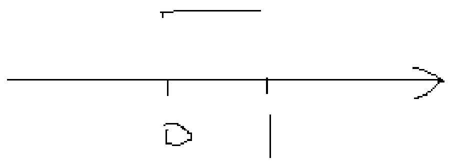
$$N_{i,1}(t) = \begin{cases} 1 & t_i < x < t_{i+1} \\ 0 & Otherwise \end{cases}$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

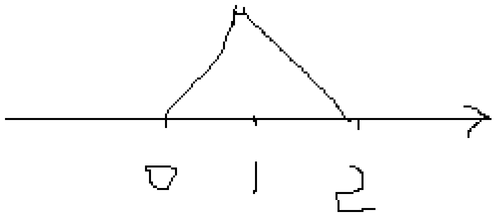
  - Knot Vector: a sequence of non-decreasing number

$$t_0, t_1, \cdots, t_{k-1}, t_k, \cdots, t_n, t_{n+1}, \cdots, t_{n+k-1}, t_{n+k}$$

- $k=1$, i = 0



- $k=2$, i = 0



$$N_{i,1}(t)=\begin{cases}1 & t_i < x < t_{i+1}\\ 0 & Otherwise\end{cases}$$

$$N_{i,k}(t)=\frac{t-t_i}{t_{i+k-1}-t_i}N_{i,k-1}(t)+\frac{t_{i+k}-t}{t_{i+k}-t_{i+1}}N_{i+1,k-1}(t)$$

B-Spline Basis Function

- Questions:
  - What is nonzero domain(非零区间) of B-Spline basis function $N_{i,k}(t)$?
  - How many knots does it need?
  - What is the definition domain(定义区间) of the curves?

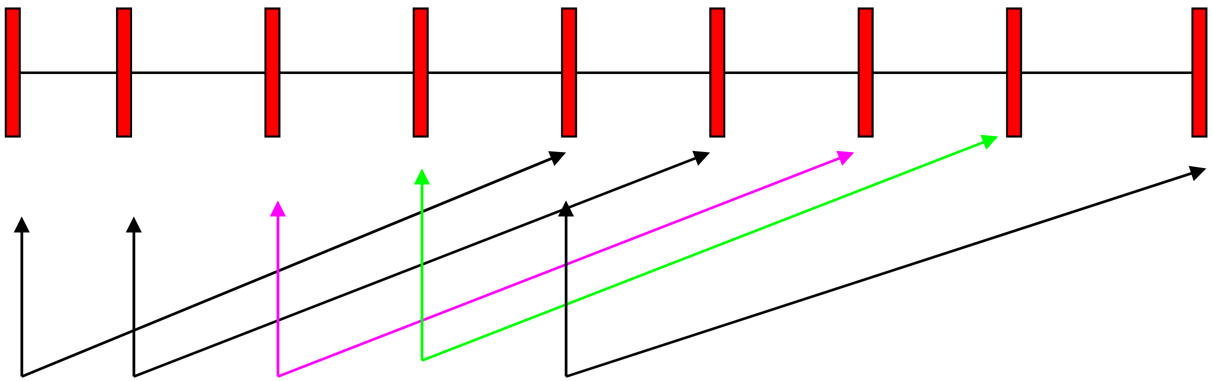
$$P(t) = \sum_{i=0}^{4} P_i N_{i,4}(t)$$

B-Spline Basis Function

- take k=4, n=4 as example

$$P(t)=\sum_{i=0}^{4} P_i N_{i,4}(t)$$

$$t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8$$



B-Spline Basis Function

- Properties:
  - Non-negativity and local support
    - $N_{i,k}(t)$ is non-negative
    - $N_{i,k}(t)$ is a non-zero polynomial on $[t_i, t_{i+k}]$

$$N_{i,k}(t)\begin{cases} \geq 0 & t \in [t_i, t_{i+k}] \\ = 0 & otherwise \end{cases}$$

  - Partition of Unity
    - The sum of all non-zero order k basis functions on $[t_{k-1}, t_{n+1}]$ is 1

$$\sum_{i=0}^{n} N_{i,k}(t) = 1 \qquad t \in [t_{k-1}, t_{n+1}]$$

B-Spline Basis Function

- Properties:
  - Differential equation of the basis function:

$$N'_{i,k}(t) = \frac{k-1}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{k-1}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

  - Please compare with the Bernstein base:

$$B'_{i,n}(t) = n[B_{i-1,n-1}(t) - B_{i,n-1}(t)],$$

$$i = 0,1,\cdots,n;$$

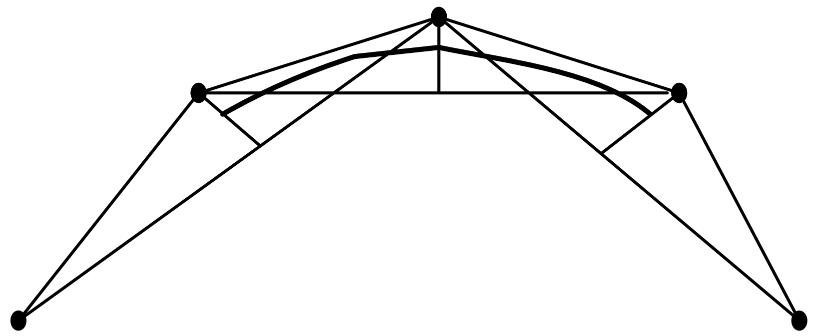
B-Spline

- Category(分类) of B-Spline
  - General Curves could be categorized to two groups by checking if the start point and the endpoint are overlapped:
    - Open Curves
    - Close Curves
  - According to the distribution(分布) of the knots in knot vector, B-Spline could be classified to the following four groups:

Uniform B-Spline(均匀B样条)

– (1) Uniform B-Spline

- The knots are uniformed distributed, like 0,1,2,3,4,5,6,7
- This kind of knot vector defines uniform B-Spline basis function

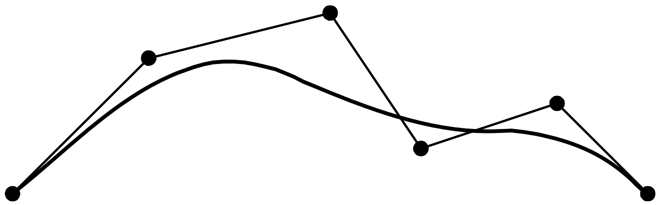


uniform B-Spline of Degree 3

Quasi-Uniform B-Spline(准均匀B样条)

– (2) Quasi-Uniform B-Spline

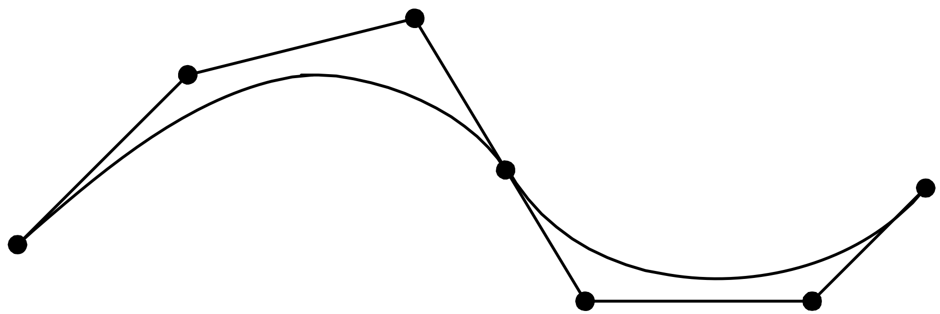
- Different from uniform B-Spline, it has:
  - the start-knot and end-knot have repetitiveness(重复度) of k
  - Uniform B-Spline does not retain the “end point” property of Bezier Curve, which means the start point and end point of uniform B-Spline are no-longer the same as the start point and end point of the control points. However, quasi-Uniform B-Spline retains this “end point” property



Quasi-uniform B-Spline curve of degree 3

Piecewise Bezier Curve(分段Bezier曲线)

– (3) Piecewise Bezier Curve
  - the start-knot and end-knot have repetitiveness(重复度) of k
  - all other knots have repetitiveness of k-1
  - Then each curve segment will be Bezier curves



Piecewise B-Spline Curve of degree 3

Piecewise Bezier Curve(分段Bezier曲线)

- For piecewise Bezier curve, the different pieces of the curve are relatively independent. Moving the control point will only influence the corresponding piece of curve, while other pieces of curves will be not change. Furthermore, the algorithms for Bezier could also be used for piecewise Bezier Curve.
- But this method need more data to define the curve (more control points, more knots).

Non-uniform B-Spline (非均匀B样条)

- (4) Non-uniform B-Spline
  - The knot vectors $T=[t_0,t_1,\cdots,t_{n+k}]$ satisfy conditions that the sequence of knots is non-decreasing(非递减).
    - repetitiveness of two end knots, <= k
    - repetitiveness of other knots, <= k-1
  - This kind of knot vector defines the non-uniform B-Spline.

Properties of B-Spline Curves

- Properties of B-Spline curves
  - Local(局部性)
    - The curve in interval $t \in [t_i, t_{i+1}]$ is only affected by at most k control points $P_j (j = i-k+1, \cdots, i)$ and is independent of other control points.
    - Changing the position of control point P_i will only affect the curve on interval (t_i, t_{i+k})

$$P(t) = \sum_{i=0}^{n} P_i N_{i,k}(t)$$

Properties of B-Spline

– Continuity(连续性)

  - P(t) is C^{k-1-r} continuous at a node of repetitiveness r.

– Convex hull(凸包性)

  - A B-spline curve is contained in the convex hull of its control polygon. More specifically, if t is in knot span $(t_i, t_{i+1}), k-1 \le i \le n$, then P(t) is in the convex hull of control points $P_{i-k+1}, \cdots, P_i$

Properties of B-Spline

– Piecewise polynomial (分段多项式)
  - In every knot span, P(t) is a polynomial of t whose degree is less than k.
– Derivative formula(导数公式)

$$P'(t) = \left(\sum_{i=0}^{n} P_i N_{i,k}(t) \right)' = \sum_{i=0}^{n} P_i N_{i,k}{}'(t)$$

$$= (k-1) \sum_{i=1}^{n} \left(\frac{P_i - P_{i-1}}{t_{i+k-1} - t_i} \right) N_{i,k-1}(t) \qquad t \in [t_{k-1}, t_{n+1}]$$

Properties of B-Spline

– Variation Diminishing Property(变差缩减性)
  - this means no straight line intersects a B-spline curve more times than it intersects the curve's control polygons.

– Geometry invariability(几何不变性)
  - The shape and position of curve are independent with the choosing of coordinate system(坐标系).

Properties of B-Spline

– Affine invariability(仿射不变性)

$$A[P(t)] = \sum_{i=0}^{n} A[P_i] N_{i,k}(t), \quad t \in [t_{k-1}, t_{n+1}]$$

- It means that the form of equation is invariable under the affine transformation.

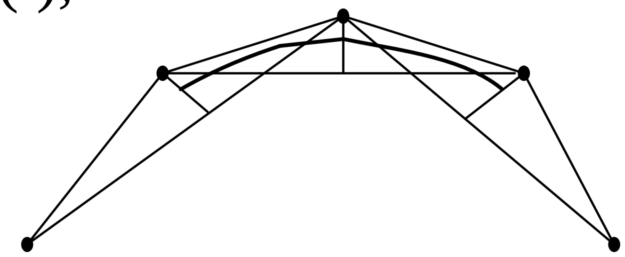
– Line holding(直线保持性)

- It means that if the control polygon degenerates to a line, the B-Spline curve also degenerates to a line.

Properties of B-Spline

- Flexibility(灵活性)
  - Using B-Spline Curve we can easily construct special cases such as line segment(线段), cusp(尖点), tangent line(切线).
  - For example, for B-Spline of order 4 (degree 3), if you want to construct a line segment, you only need to specify $P_i, P_{i+1}, P_{i+2}, P_{i+3}$ collinear (共线).
  - If you want the curve pass point P(i), you only need to let

$$P_i = P_{i+1} = P_{i+2}$$



Properties of B-Spline

- If you want the curve tangent(相切) to a specific line L, you only need to specify P_i, P_{i+1}, P_{i+2} on line L, and repetitiveness of t_{i+3} less than 2.

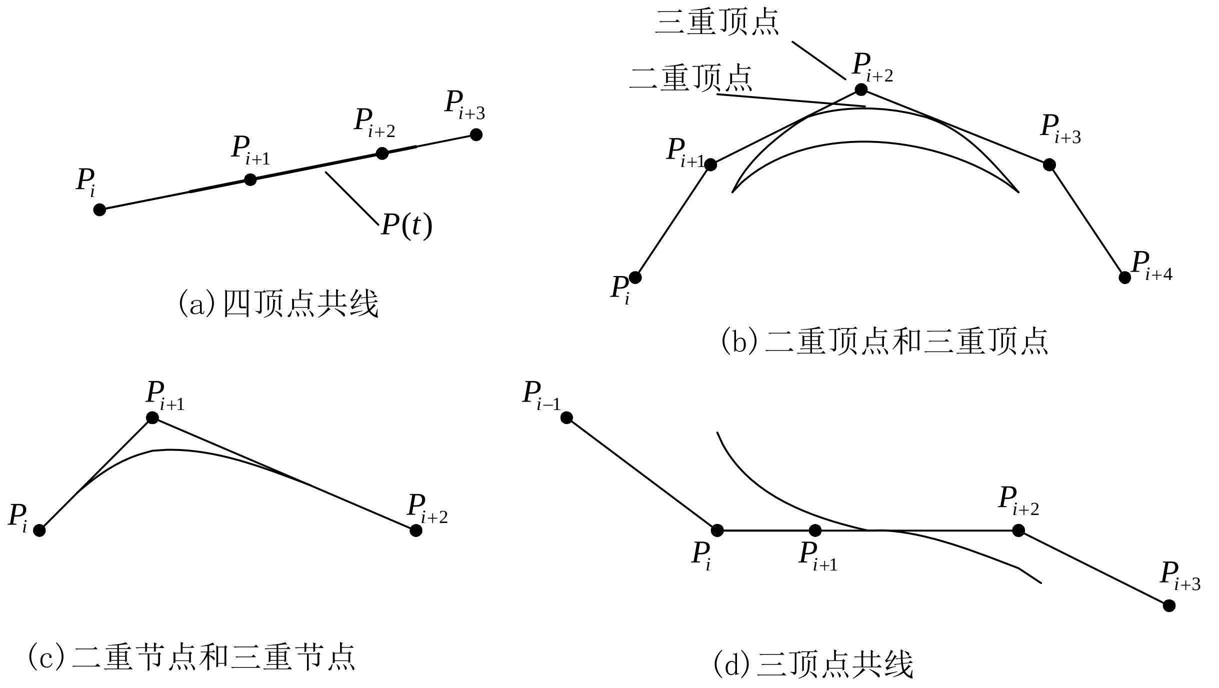


图.1.26 三次B样条曲线的一些特例

De Boor Algorithm

- To compute P(t) (a point on the curve), you could use B-Spline formula, but it is more efficient to use de Boor algorithm.
- De Boor Algorithm:

$$P(t) = \sum_{i=0}^{n} P_i N_{i,k}(t) = \sum_{i=j-k+1}^{j} P_i N_{i,k}(t)$$

$$= \sum_{i=j-k+1}^{j} P_i \left[\frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t) \right]$$

$$= \sum_{i=j-k+1}^{j} \left[\frac{t - t_i}{t_{i+k-1} - t_i} P_i + \frac{t_{i+k-1} - t}{t_{i+k-1} - t_i} P_{i-1} \right] N_{i,k-1}(t) \qquad t \in [t_j, t_{j+1}]$$

De Boor Algorithm

- Let

$$P_i^{[r]}(t) = \begin{cases} P_i, r = 0, i = j-k+1, j-k+2, \cdots, j \\ \dfrac{t - t_i}{t_{i+k-r} - t_i} P_i^{[r-1]}(t) + \dfrac{t_{i+k-r} - t}{t_{i+k-r} - t_i} P_{i-1}^{[r-1]}(t), \\ \qquad r = 1, 2, \cdots, k-1; i = j-k+r+1, j-k+r+2, \cdots, j \end{cases}$$

- Then

$$P(t) = \sum_{i=j-k+1}^{j} P_i N_{i,k}(t) = \sum_{i=j-k+2}^{j} P_i^{[1]}(t) N_{i,k-1}(t)$$

- This is De Boor Algorithm.

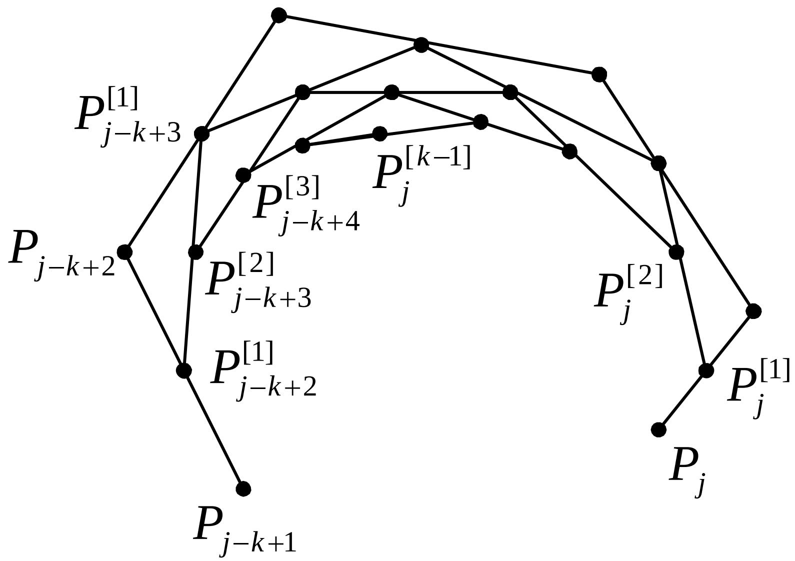
De Boor Algorithm

- The recursion of De Boor Algorithm is as shown below:

$$
\begin{array}{ccccccc}
P_0 & & & & & & \\
P_1 & & & & & & \\
\vdots & & & & & & \\
P_{j-k+1} & & & & & & \\
P_{j-k+2} & \rightarrow & P_{j-k+2}^{[1]} & & & & \\
P_{j-k+3} & \rightarrow & P_{j-k+3}^{[1]} & \rightarrow & P_{j-k+3}^{[2]} & & \\
\vdots & & \vdots & & \vdots & \vdots & \\
P_j & \rightarrow & P_j^{[1]} & \rightarrow & P_j^{[2]} & & P_j^{[k-1]} \\
\vdots & & & & & & \\
P_n & & & & & &
\end{array}
$$

De Boor Algorithm

- Geometric meaning (几何意义) of De Boor Algorithm
  - It has an intuitive geometric interpretation: corner cutting (割角).
    - It means that using line segment $P_i^{[r]}P_{i+1}^{[r]}$ to cut corner $P_i^{[r-1]}$. Begin from polygon $P_{j-k+1}P_{j-k+2}\cdots P_j$, after k-1 steps of cutting, we finally get point $P_j^{[r-1]}(t)$ on the curve P(t).



Knot Insertion

- Knot insertion
  - An important tool for practical interactive use of B-spline, which allows one to add a new knot to a B-spline without changing the shape or its degree. For instance, one may want to insert additional knots in order to be able to raise flexibility of shape control.
    - Insert a new knot t to a knot span $[t_i, t_{i+1}]$
    - The knot vector becomes:

$$T^1 = [t_0, t_1, \cdots, t_i, t, t_{i+1}, \cdots, t_{n+k}]$$

    - denoted as: $T^1 = [t_0^1, t_1^1, \cdots, t_i^1, t_{i+1}^1, t_{i+2}^1, \cdots, t_{n+k+1}^1]$

Knot Insertion

- The new knot sequence defines a new group of B-Spline Basis Functions. The original curve *P(t)* can be expressed by this new group of Basis Functions and new control points P_j^1 , which are unknown.

$$P(t) = \sum_{j=0}^{n+1} P_j^1 N_{j,k}^1(t)$$

Knot Insertion

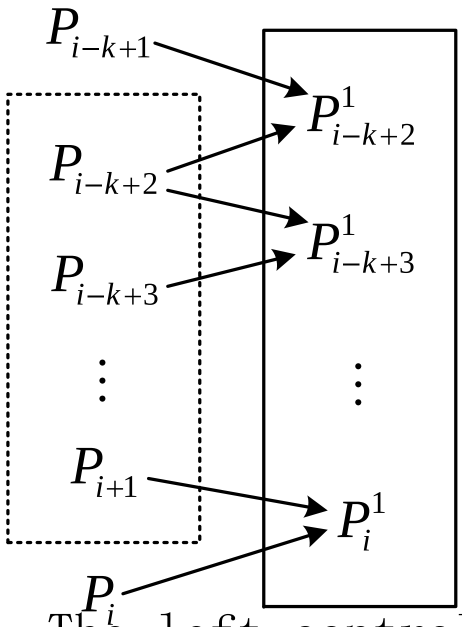
- Boehm gives a formula for computing new control points:

$$
\begin{cases}
P_j^1 = P_j, & j = 0,1,\cdots,i-k+1 \\
P_j^1 = (1-\beta_j)P_{j-1} + \beta_j P_j, & j = i-k+2,\cdots,i-r \\
P_j^1 = P_{j-1}, & j = i-r+1,\cdots,n+1
\end{cases}
$$

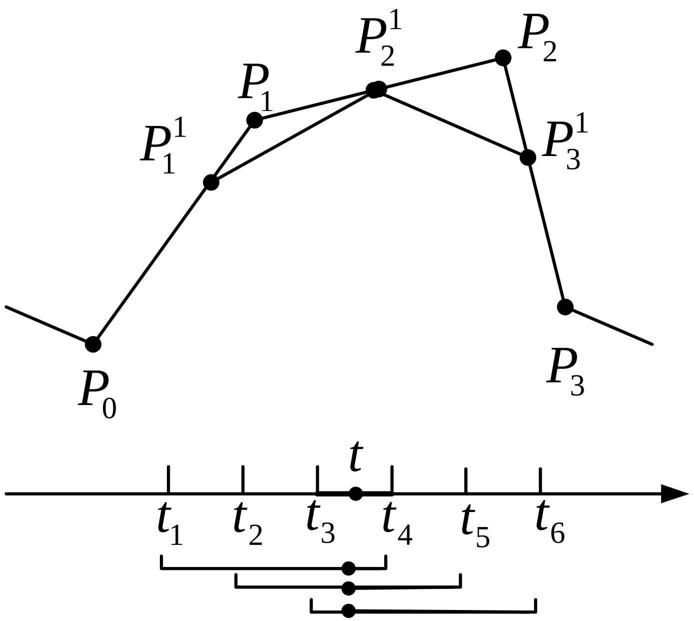
$$
\beta_j = \frac{t - t_j}{t_{j+k-1} - t_j}
$$

- r is the repetitiveness of newly inserted knot *t* in the knot sequence.

Knot Insertion



The left control points(totally k-1 points) are replaced by the right ones(totally k points)



Insert $t \in [t_3, t_4]$

See demo of knot insertion

B-Spline Surface(B样条曲面)

- Given knot vectors in axes(参数轴) U and V:

$$U = [u_0, u_1, \cdots, u_{m+p}]$$
$$V = [v_0, v_1, \cdots, v_{n+q}]$$

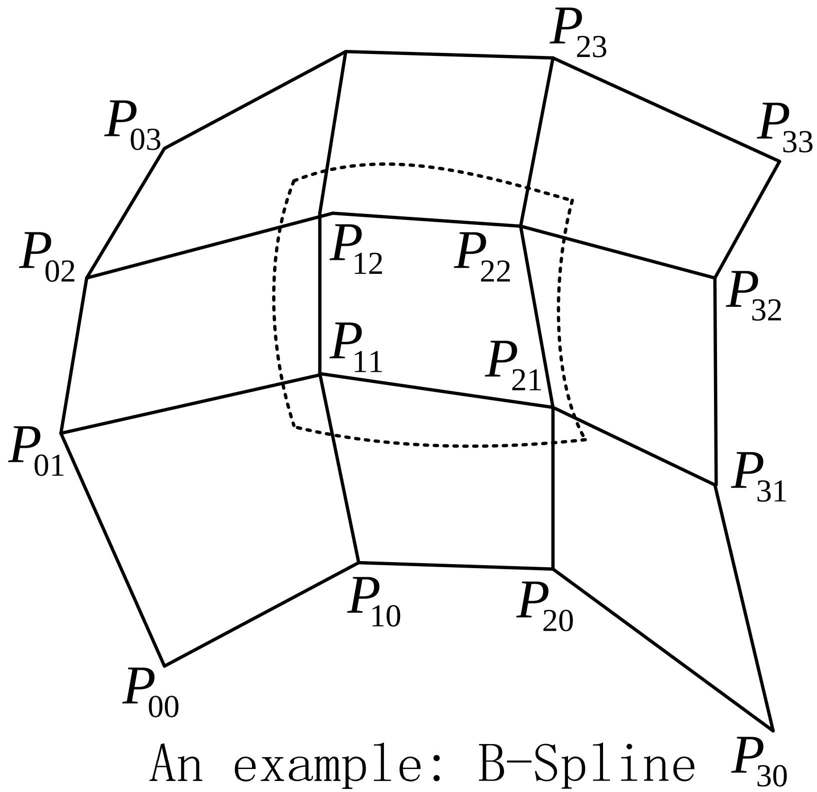
- B-Spline surface of order p×q is defined as:

$$P(u,v) = \sum_{i=0}^{m} \sum_{j=0}^{n} P_{ij} N_{i,p}(u) N_{j,q}(v)$$

B-Spline Surface

- P_{ij} are the control points of B-Spline Surface, which are usually referred to as the control net (控制网格、特征网格).
- $N_{i,p}(u)$ and $N_{j,q}(v)$ are B-Spline basis functions of order p and q, one for each direction (U and V), which could also be computed by de Boor-Cox formula.

B-Spline Surface



An example: B-Spline surface of degree 3*3(双三次B样条曲面片)

NURBS Curve and Surface

- Disadvantage of B-Spline curve and Bezier Curve:
  - can’t accurately represent conic(圆锥曲线) curve except parabola(抛物线),
- NURBS (Non-Uniform Rational B-Spline) (非均匀有理B样条)
  - In order to find a mathematical method that could represent conic and conicoid(二次曲面) accurately.

NURBS

- NURBS is too complex
- Les Piegl's The NURBS Book,
  - "NURBS from Projective Geometry to Practical Use"
  - Les Piegl， Graduated from Budapest University of Hungary, many years in SDRC company for geometric modeler design

NURBS

Some years ago a few researchers joked about NURBS, saying that the acronym really stands for NOBODY Understands Rational B-Splines, write the authors in their foreword; they formulate the aim of changing NURBS to EURBS, that is, Everybody….

There is no doubt that they have achieved this goal....

I highly recommend the book to anyone who is interested in a detailed description of NURBS. It is extremely helpful for students, teachers and designers of geometric modeling systems. —— *Helmut Pottmann*

NURBS

- Advantages of NURBS
  - It provide a general and accurate representation for representing and designing free curves or surfaces(自由曲线曲面)
  - They offer one common mathematical form for both standard analytical shapes (e.g., conics) and free-form shapes (parametric form).
  - can be evaluated reasonably fast by numerically stable and accurate algorithms;

NURBS

- Advantages of NURBS
  - They are invariant under affine as well as perspective transformations.
  - the control points and weights can be modified, which gives great flexibility for designing curves/surfaces.
  - non-rational B-Spline, non-rational and rational Bezier could be viewed as special cases of NURBS.

NURBS

- Some difficult problems in using NURBS
  - Require more storage than traditional curve/spline methods. (such as circle representation)
  - If the weights are set inappropriately, the curve will be abnormal(畸变).
  - It is very complex to deal with situations like curve overlap(曲线重叠).

NURBS

- Before discussing about NURBS, let's first review the definition of B-Spline:

$$P(t) = \sum_{i=0}^{n} P_i N_{i,k}(t)$$

$$N_{i,k}(t) = \frac{t - t_i}{t_{i+k-1} - t_i} N_{i,k-1}(t) + \frac{t_{i+k} - t}{t_{i+k} - t_{i+1}} N_{i+1,k-1}(t)$$

$$t_0, t_1, \cdots, t_{k-1}, t_k, \cdots, t_n, t_{n+1}, \cdots, t_{n+k-1}, t_{n+k}$$

NURBS

- Definition of NURBS curve.
  - NURBS Curves are defined by piecewise rational B-Spline polynomial basis function (分段有理B样条多项式基函数):

$$P(t)=\frac{\sum_{i=0}^{n}\omega_i P_i N_{i,k}(t)}{\sum_{i=0}^{n}\omega_i N_{i,k}(t)}=\sum_{i=0}^{n}P_i R_{i,k}(t)$$

$$R_{i,k}(t)=\frac{\omega_i N_{i,k}(t)}{\sum_{j=0}^{n}\omega_j N_{j,k}(t)}$$

Definition of NURBS

- NURBS basis $R_{i,k}(t)$ retains all properties of B-Spline Basis.
  - Local support: Ri,k(t)=0， t$\notin$[ti, ti+k]
  - Partition of Unity: ⟶ $$\sum_{i=0}^{n} R_{i,k}(u) = 1$$
  - Differentiability(可微性):
    - If *t* is not a knot, *P(t)* is infinitely differentiable(无限次可微) in the knot interval. If *t* is a knot, *P(t) is only* C-(k-r) continuous.
  - If ω_i=0, then $R_{i,k}(t)$=0;
  - If ω_i=+∞, then $R_{i,k}(t)$=1;

Definition of NURBS

- NURBS curve has similar geometric properties as the B-Spline Curve:
  - Local support (局部支持性).
  - Variation Diminishing Property(变差缩减性)
  - Strong Convex hull(凸包性)
  - Affine invariability(仿射不变性)
  - Differentiability(可微性)
  - If the weight of a control point is 0, then corresponding control point doesn’t affect the curve.

Definition of NURBS

- If $\omega_i \to \infty$, and $t \in [t_i, t_{i+k}]$, then $P(t) = P_i$
- Non-rational/rational Bezier curves and non-rational B-Spline curves are special cases of NURBS curve.

Homogenous Coordinates(齐次坐标)

- Representation in Homogeneous coordinates
  - in homogeneous coordinates system *xyw*, control point is represented as:

$$P_i^{\omega} = (\omega_i x_i, \omega_i y_i, \omega_i), \quad (i = 0,1,\cdots,n)$$

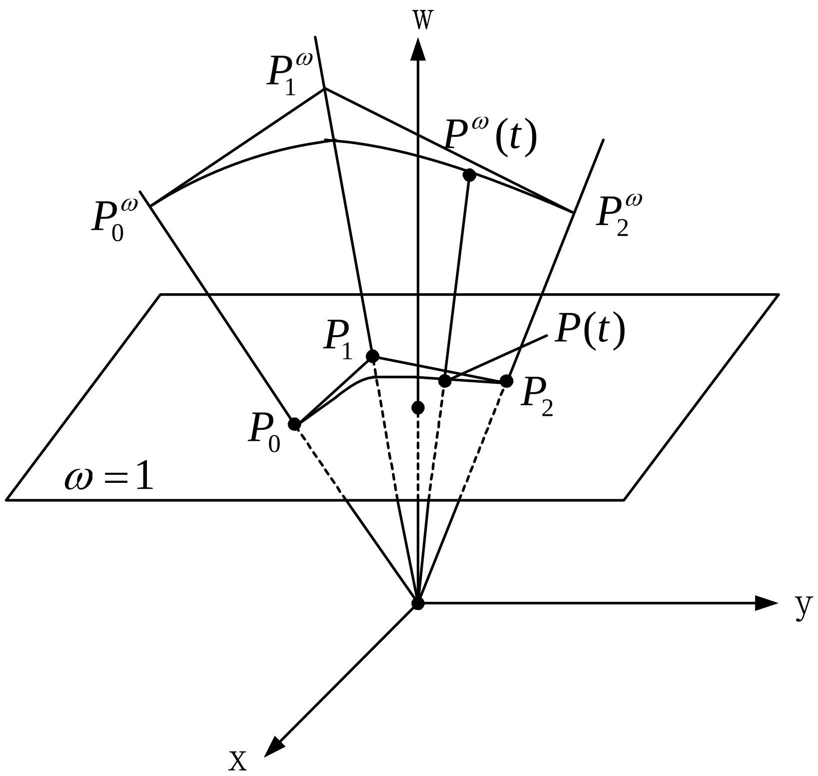
  - non-rational B-Spline curve of order k:

$$P^{\omega}(t) = \sum_{i=0}^{n} P_i^{\omega} N_{i,k}(t)$$

Homogenous Coordinates

– By projecting the curve to the origin(坐标原点),we can get a planar curve(平面曲线).

$$P(t) = \frac{\sum_{i=0}^{n} \omega_i P_i N_{i,k}(t)}{\sum_{i=0}^{n} \omega_i N_{i,k}(t)}$$



Homogenous Coordinates

– NURBS curve in 3D space could be similar defined as:

$$P_i^{\omega} = (\omega_i x_i, \omega_i y_i, \omega_i z_i, \omega_i), \quad (i = 0, 1, \cdots, n)$$

– the algorithms for non-rational B-Spline curve could also be used for NURBS curve, if represented in homogenous coordinates.

Geometric meaning of weights

- Geometric meaning of weights
  - If fix parameter *t*, and let weight ω_i change, then NURBS curve equation *P(t)* becomes a linear equation of ω_i, which means that all the points with the same *t* are collinear(共线)

Geometric meaning of weights

- B, N, B_i are points on the curve with $\omega_i = 0, \omega_i = 1, \omega_i \neq 0{,}1$

  where $B = P(t;\omega_i = 0)$, $N = P(t;\omega_{i=1}), B_i = P(t;\omega_i \neq 0,1)$
- we define $P_i = P(t;\omega_i \to \infty)$
- N, B_i can be rewritten as:

$$N = (1-\alpha)B + \alpha P_i$$

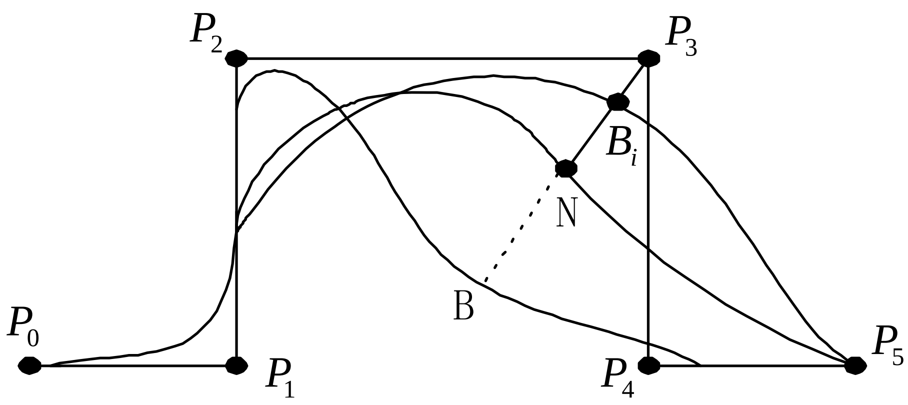
$$B_i = (1-\beta)B + \alpha P_i$$

- We have:

$$\frac{1-\alpha}{\alpha} : \frac{1-\beta}{\beta} = \frac{P_i N}{BN} : \frac{P_i B_i}{BB_i} = \omega_i$$

Geometric meaning of weights

- If ω_i increases/decreases, β also increases/decreases, and the curve is like pulling to/pushing away from point P_i.



NURBS representation of Conic(圆锥曲线)

- NURBS representation of Conic
  - For knot vectors $T=[0,0,0,1,1,1]$, NURBS curve degrades to a degree 2 Bezier curve, and it could be easily proved that this curve is conic :

$$P(t)=\frac{(1-t^2)\omega_0 P_0+2t(1-t)\omega_1 P_1+t^2\omega_2 P_2}{(1-t)^2\omega_0+2t(1-t)\omega_1+t^2\omega_2}$$

  - $C_{sf}=\dfrac{\omega_1^2}{\omega_0\omega_2}$ is the geometry factor(形状因子).
  - C_{sf} decides type of the conic curve.
  - If $C_{sf}=1$, the curve is a parabola(抛物线).

NURBS representation of Conic

- When $C_{sf} \in (1,+\infty)$, it is a hyperbolic arc(双曲线)
- When $C_{sf} \in (0,1)$, it is a elliptic arc(椭圆)
- When $C_{sf} = 0$, it degenerates to a couple of line P_0P_1 and P_1P_2.
- When $C_{sf} \to +\infty$, it degenerates to a line P_0P_2.

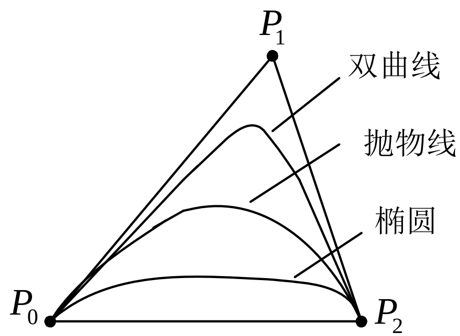
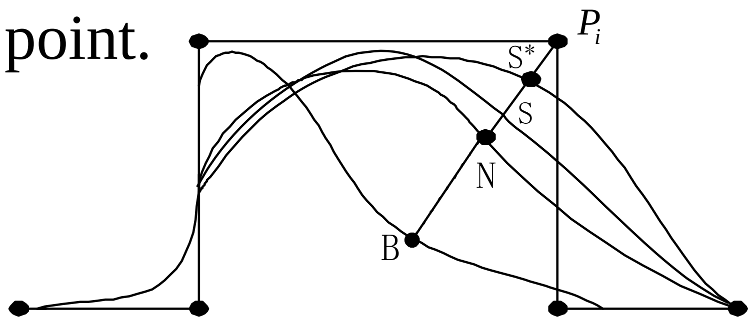


图3.1.36 圆锥曲线的 NURBS表示

Modification

- Modification of NUBRS curve.
  - The most important modification includes:
    - modify weight, control points, and knots.
  - Modify weight.
    - Decrease/increase the weight of a control point (control points and other weights are fixed), the curve is like being pushed away or pulled to the corresponding control point.



Modification

- If want a point S on the curve becomes away from/close to control point *Pi* for distance *d* (get a new point S'), the newly weight ω^* is computed as:

$$\omega^* = \omega_i \left[1 + \frac{d}{R_{i,k}(t)(P_i S - d)} \right]$$

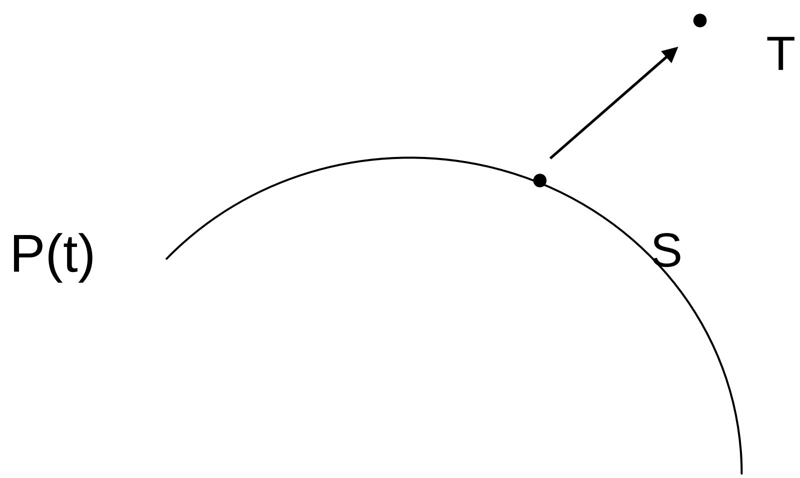
– Modify control point:

- The shape of curve changes if changing the position of a control point.

Modification

– Shape modification under geometric constraints (几何约束下的形状修改)
  - Formulation:

    How to compute the new control vertex?
    to make the point S on the curve change to a new point T.



Modification

– the curve equation is rewritten as:

$$P(t) = \sum_{i=0}^{n} P_i R_{i,k}(t), \quad t_{k-1} \le t \le t_{n+1}$$

Where,

$$R_{i,k}(t) = \frac{W_i N_{i,k}(t)}{\sum_{i=0}^{n} W_i N_{i,k}(t)}$$

Modification

– Constrained optimization method(带约束的优化方法)

- Suppose the control point $P_l, P_{l+1}, \cdots P_{l+m-1}$ be changed. Give each control point a small change $\varepsilon_i = (\varepsilon_i^x, \varepsilon_i^y, \varepsilon_i^z)^T$ Then use the constrained optimization method to compute it. The constraint will be:

$$\tilde{P}(t) = \sum_{i=0}^{l-1} P_i R_{i,k}(t) + \sum_{i=l}^{l+m-1} (P_i + \varepsilon_i) R_{i,k}(t) + \sum_{i=l+m}^{n} P_i R_{i,k}(t)$$

$$= \sum_{i=0}^{n} P_i R_{i,k}(t) + \sum_{i=l}^{l+m-1} \varepsilon_i R_{i,k}(t) \qquad t_{k-1} \le t \le t_{n+1};$$

Modification

- Let $$\sum_{i=l}^{l+m-1} \left\| \varepsilon_i \right\|^2 = \text{Min}$$
- Duo to Lagrange function:

$$L = \sum_{i=l}^{l+m-1} \left\| \varepsilon_i \right\|^2 + \lambda \left(T - \tilde{P}(t_s) \right)$$

- We get an equation system:

$$\begin{cases} T = S + \sum_{i=l}^{l+m-1} \varepsilon_i R_{i,k}(t_s) \\ \varepsilon_i = \dfrac{\lambda}{2} R_{i,k}(t_s), \qquad i = l, l+1, \cdots, l+m-1 \end{cases}$$

Modification

- those equations could be solved as:

$$\varepsilon_i = \frac{R_{i,k}(t_s)}{\sum_{j=l}^{l+m-1} R_{j,k}^2(t_s)}(T-S), \quad i = l, l+1, \cdots, l+m-1$$

- If only one control point is allowed to be modified, we have:

$$\varepsilon = \frac{T-S}{R_{i,k}(t_s)}$$

This formula was proposed by Piegl(editor in chief of CAD) in 1989

- This method could also be easily extended to other geometric constraints and surfaces case.

Modification

– We can also Energy Minimization for shape modification

- Curve: Strain energy

$$E(p) = \int k^2 ds = \int_0^1 \left(\frac{|P' \times P''|}{|P'|^3} \right)^2 dt \longrightarrow E(p) = \int |P''|^2 dt$$

- Surface: Thin plate energy

$$E(P) = \iint (P_{uu}^2 + 2P_{uv}^2 + P_{vv}^2) dudv,$$

Hu Shi-Min, etal., Modifying the shape of NURBS surfaces with geometric constraints, CAD, 2001, Vol. 33, No. 12, 903-912.

NURBS surface

- Definition of NUBRS surface

$$P(u,v)=\frac{\sum_{i=0}^{m}\sum_{j=0}^{n}\omega_{ij}P_{ij}N_{i,p}(u)N_{j,q}(v)}{\sum_{i=0}^{m}\sum_{j=0}^{n}\omega_{ij}N_{i,p}(u)N_{j,q}(v)}=\sum_{i=0}^{m}\sum_{j=0}^{n}P_{ij}R_{i,p;j,q}(u,v)\quad u,v\in[0,1]$$

$$R_{i,p;j,q}(u,v)=\frac{\omega_{ij}N_{i,p}(u)N_{j,q}(v)}{\sum_{r=0}^{m}\sum_{s=0}^{n}\omega_{rs}N_{r,p}(u)N_{s,q}(v)}$$

NURBS Surface

- Weight of four corners are positive:

$$\omega_{00}, \omega_{m0}, \omega_{0n}, \omega_{mn} > 0$$

- weight of other points are non-negative:

$$\omega_{ij} \geq 0$$

- Properties of NURBS surface:

  $R_{i,p;j,q}(u,v)$ has similar properties as the non-rotational B-Spline:

  - local support
  - partition of unity

NURBS Surface

- Differentiability: At a knot with repetitiveness k
  - If it's in direction *U* , it's C-(p-r-1) continuous differentiable.
  - If it's in direction *V* , it's C-(q-r-l) continuous differentiable.
- Local extremum(极值):
  - If p,q>1, there also exists a loal maximum(极大值).

Thank you very much!

选作大实验-半透明物体绘制

- 要求
  – 绘制带有半透明效果的物体
  – 选作该实验可以替代光线跟踪的大实验

半透明材质效果

半透明

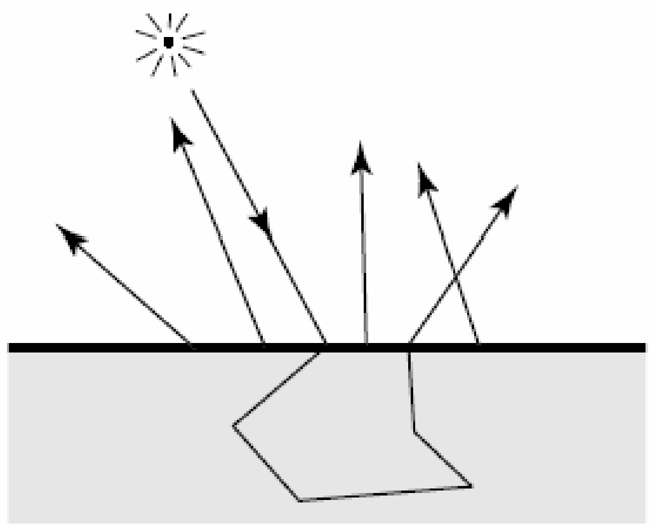
不透明

半透明材质效果

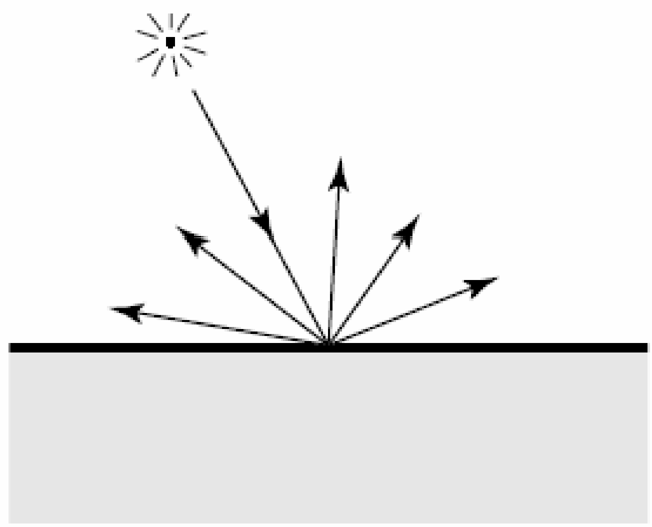
半透明

不透明

基本原理



半透明
次表面散射



不透明
仅在表面反射

后续安排

- 请有兴趣的同学可发邮件给助教(徐昆,xu-k@mails.tsinghua.edu.cn 或 程明明, cmm07@mails.tsinghua.edu.cn)报名。
- 确定参加人数之后，将给选作同学集中安排一个talk，介绍半透明绘制算法。

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- 孙家广 胡事民 计算机图形学基础教程